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MATHEMATICAL PROBLEMS
OF
RADIATIVE EQUILIBRIUM

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MATHEMATICAL PROBLEMS

OF

RADIATIVE EQUILIBRIUM

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CONTENTS

CHAPTER III

DISCUSSION OF PROBLEMS III AND IV

CHAPTER IV

EXPLICIT SOLUTION OF CERTAIN INTEGRAL EQUATIONS

CHAPTER V

OTHER PROBLEMS OF RADIATIVE EQUILIBRIUM

INTRODUCTION

K Schwarzschild's classical work on absorption and diffusion in the sun's atmosphere, and the continuation of these investigations through A S. Eddington, J H Jeans, E A Milne and others, have rendered the theory of radiative equilibrium a definite chapter of mathematical astrophysics The problems connected with this theory are of the following type. Gaseous material filling a certain part of space is subject to given incident radiation.* What is the distribution of light and of temperature in the medium, when in radiative equilibrium? In the outer layers of a star, where the curvature can be neglected, i e where the material can be considered as stratified in parallel planes, those problems appear in their simplest mathematical form. The astrophysicists mostly contented themselves with an approximate solution of these problems Owing to a certain inherent beauty, however, they aroused also the interest of the rigorous mathematician It is the purpose of this tract to attempt a coherent representation of all that has been achieved in the direction of a rigorous treatment of those standard problems.

The rigorous solution of a physical problem naturally presupposes an exact formulation of the physical assumptions implied in the problem. Has then the rigorous solution more than merely mathematical significance, when these assumptions prove right only with a limited degree of approximation? The answer to this question is not always in the negative. The Schuster-Schwarzschild model of a purely scattering atmosphere, or the Schwarzschild model of a purely absorbing gray atmosphere, both in radiative equilibrium, represent for instance typical standard models which play a similar rôle in the theory of the stellar atmospheres as the 'intermediate orbits' in celestial mechanics. The Milne model, being of special importance as a

* More generally, on the boundary the radiation satisfies given conditions.

simple limit case (infinite optical depth) of Schwarzschild's model, may well be compared with Hill's periodic orbit in the lunar theory. It is well known what the detailed study of an intermediate orbit means for the computation of the actual one. A more detailed and rigorous treatment of the above standard models of radiative equilibrium should thus deserve more than mere mathematical interest.

Besides the known results (theory of Milne's standard model, infinitely deep slab with no radiation incident on its surface) the reader will also find more or less detailed discussions of other models. Milne's model with radiation incident on the boundary (insolation of a planetary atmosphere, reflexion effect in close binary stars), the Schuster-Schwarzschild model with an arbitrary law of scattering, narrower limits for the solution of the Schuster-Schwarzschild standard problem, Milne's model of a planetary nebula, and other problems.

From the mathematical point of view, the main feature of the above models is that they are governed by integral equations, which are linear in the standard cases (pure absorption and gray material, pure scattering). Its important property, namely the positivity of the kernel, has already been recognized by Schwarzschild as fundamental for the rigorous discussion. It may be mentioned that the reading of the book requires no special knowledge of the theory of integral equations, since the positivity of the kernel allows all problems to be treated in an elementary way. Only Chapter IV presupposes some knowledge of the Fourier integral.

It is a pleasure to the author to thank Prof. Milne for many helpful remarks and the Syndics of the University Press for accepting this book as a Tract.

E. H.

Nov. 1, 1933

CHAPTER I

FUNDAMENTAL PRINCIPLES AND PROBLEMS

§ 1 TRANSMISSION OF RADIATION THROUGH ABSORBING AND SCATTERING MATERIAL

The outer layers of a star are constantly exposed to an enormous flow of radiant energy coming from the deep interior In working itself through the outer layers, a part of the radiation is absorbed and scattered. The scattered radiation will immediately be redistributed in the different directions, the frequency being unchanged On the other hand, a particle heated up by the absorption of radiation re-emits temperature radiation of all frequencies

As a first approximation, the state of matter and radiation may be considered stationary Therefore we must first find the conditions for a steady state The intensity of radiation changes in a definite way along the ray A part will be lost through absorption and scattering On the other hand, the intensity gains again since the particles on the ray emit scattered and temperature radiation in the direction of the ray This law of transfer holds separately for each frequency.

Combined with the condition for a steady state the energy principle yields another equation The net loss of radiant energy within an element of volume (radiation of *all* frequencies) equals the net gain of heat energy through conduction and convection plus the radiant energy liberated from other (sub-atomic) sources. Energy of the latter kind must play a rôle in the interior of the stars in order to maintain their radiation over cosmical ranges of time.

When energy is transferred by radiation only, the net flow of radiation through each volume element vanishes. This state of affairs is usually called radiative equilibrium (in the strict sense). Modern astrophysics claims the outer layers of a star to be approximately in radiative equilibrium. In the stellar interior

the energy liberated is no more negligible though being small in comparison to the radiant energy itself In spite of this fact it has become customary to speak of radiative equilibrium (in the wider sense) in this case

For the determination of the temperature distribution, one must know the laws relating the temperature of matter to the radiation When matter and radiation are in an enclosure, Kirchhoff's and Plank's laws hold. In the stars, however, the temperature varies from layer to layer, so that those laws cannot be strictly valid In the interior, however, the radiation is practically enclosed, and even in the outer layers those laws have proved to be a very good approximation.

In the steady state all quantities describing the state are independent of time The radiation field is described by the *intensity of radiation*

$$I_\nu(P, r)$$

as a function of the point P, the direction r and the frequency ν.* Its physical meaning is the following. Let $d\sigma$ be an element of surface about P with the (sensed) normal n, and let $d\omega$ be an infinitesimal bundle of directions containing r (surface element on the direction sphere) If we write

$$dq = |\cos(n, r)|\, d\sigma \qquad \ldots .(1)$$

for the projection of $d\sigma$ upon a plane perpendicular to r, the expression

$$I_\nu(P, r)\, dq\, d\omega\, d\nu \qquad \ldots . (2)$$

represents the radiant energy of the spectral interval $d\nu$ which, per unit time, flows through $d\sigma$ and spreads out in the solid angle $d\omega$. Here is meant, the bundle of all rays through all points of $d\sigma$ and in all directions contained in $d\omega$ dq is evidently the perpendicular cross-section at P of the bundle It is convenient to have a special name for the quantity

$$I_\nu(P, r)\, dq\, d\nu \qquad \ldots\ldots(3)$$

We call it the *strength* at P of the parallel bundle of the spectral interval $d\nu$, of direction r and with the cross-section dq.

We set $\quad d\sigma\, d\nu\, \mathfrak{F}_{\nu, n} = d\sigma\, d\nu \int I_\nu(P, r) \cos(n, r)\, d\omega, \qquad \ldots\ldots(4)$

* The degree of polarization is not analysed here.

where the integral is extended through all directions r, and where n is a given direction. Let $d\sigma$ be an orientated surface element about P and with n as its positive normal. The quantity $\mathfrak{F}_{\nu,n}\,d\sigma\,d\nu$ represents then the net flux of radiant energy (of the spectral interval $d\nu$) through $d\sigma$ and per unit time. Energy flowing through $d\sigma$ from its negative towards its positive side ($\cos > 0$) is here reckoned positive, negative, however, in the opposite directions r ($\cos < 0$). Let x, y, z be three orthogonal directions, and let us write α, β, γ for the three direction cosines of the normal n with respect to x, y, z. The equation

$$\cos(n, r) = \alpha \cos(r, x) + \beta \cos(r, y) + \gamma \cos(r, z)$$

shows that we have

$$\mathfrak{F}_{\nu,n} = \cos(n, x)\,\mathfrak{F}_{\nu,x} + \cos(n, y)\,\mathfrak{F}_{\nu,y} + \cos(n, z)\,\mathfrak{F}_{\nu,z},$$

i.e. that $\mathfrak{F}_{\nu,n}$ is the component in the direction n of a vector $\mathfrak{F}_\nu$. This vector $\mathfrak{F}_\nu(P)$ is briefly called the *net flux* at P of the ν-radiation.

In a vector field we have the Gauss integral identity

$$\int_S \mathfrak{F}_{\nu,n}\,d\sigma = \int_V \operatorname{div} \mathfrak{F}_\nu\,dv, \qquad \text{.......(5)}$$

where V is a volume, S its surface, and n the outward normal of the surface element $d\sigma$.

For the total radiation of all frequencies we set

$$I = \int_0^\infty I_\nu\,d\nu, \quad \mathfrak{F} = \int_0^\infty \mathfrak{F}_\nu\,d\nu \qquad \text{.....(6)}$$

We now come to the quantities that describe the interaction between matter and radiation. Let $\eta_\nu(P)$ be the *mass coefficient of emission* at P and within $d\nu$. The quantity

$$\eta_\nu(P)\,dm\,d\omega\,d\nu \qquad \text{......(7)}$$

represents then the radiation emitted, per unit time, by the mass element dm at P, within $d\nu$ and within the solid angle $d\omega$. In analogy to the above we find it convenient to call

$$\eta_\nu(P)\,dm\,d\nu \qquad \text{......(8)}$$

the strength of a parallel bundle emitted by dm within $d\nu$ For the total emission of all frequencies we set

$$\eta(P) = \int_0^\infty \eta_\nu(P)\,d\nu \qquad \dots \quad (9)$$

Let, furthermore, $\alpha_\nu(P)$ denote the *mass coefficient of absorption* for the ν-radiation at P Along a short path ds, the amount $\rho\alpha_\nu I_\nu ds$ of the intensity I_ν is absorbed, ρ being the density of matter at P Outside of astrophysics it is more customary to introduce the linear coefficient of absorption, $\rho\alpha_\nu$

The *mass coefficient of scattering* $\sigma_\nu(P)$ is defined in an analogous way Scattering weakens the intensity I_ν by the amount $\rho\sigma_\nu I_\nu ds$ along ds $\rho\sigma_\nu$ is the linear coefficient of scattering The radiation lost by scattering at P will immediately be redistributed among the different direction r' issuing from P This distribution is described by a *law of scattering*

$$\frac{1}{4\pi}\gamma_\nu(P, r, r'), \qquad \dots\dots(10)$$

where r denotes the direction of the incident ray We have, of course,

$$\int \gamma_\nu(P, r, r')\,d\omega' = 4\pi \qquad \dots\dots \quad (11)$$

The law of scattering is supposed to have the reciprocity property

$$\gamma_\nu(P, r, r') = \gamma_\nu(P, r', r) \qquad \dots\dots(11')$$

and

$$\gamma_\nu(P; -r, r') = \gamma_\nu(P, r, r'), \qquad \dots\,.(11'')$$

where $-r$ denotes the direction opposite to r, i e the same amount of the scattered radiation is sent into r and into $-r$. These conditions are fulfilled under very general assumptions about the material The simplest case $\gamma_\nu \equiv 1$ (uniform scattering) has hitherto found the chief attention of the astrophysicists. It is, however, necessary to consider more general laws too, for instance Rayleigh's law

$$\gamma_\nu = \gamma(r, r') = \tfrac{3}{4}\{1 + \cos^2(r, r')\}$$

It is seen to fulfil (11′) and (11″), as does also every law for which γ_ν is a function of $|\cos(r, r')|$.

All quantities (except the net flux) are non-negative, I_ν, η_ν, α_ν, σ_ν, $\gamma_\nu \geqq 0$

§ 2 ABSORPTION AND SCATTERING OF THE MASS ELEMENT THE EQUATION OF TRANSFER

We now compute the ν-radiation absorbed and scattered within a mass element dm at a point P For this purpose let us first consider the parallel bundle of all rays through dm which have a given direction r. It is convenient to imagine dm subdivided into many thin columns of direction r Let ds be the length, dq the cross-section of such a column. The strength $I_\nu(P, r)\, dq d\nu$ of the parallel bundle of direction r through that column, i e. through dq, is, in consequence of absorption within the column, weakened by the amount $\rho\sigma_\nu I_\nu dq d\nu ds$ Since $\rho dq ds$ is the mass of the column, we infer by summation over all columns that

$$\sigma_\nu I_\nu dm d\nu$$

is the amount that the strength of the bundle considered above loses on account of absorption within dm By strength of that bundle is meant here the sum of the strengths of all the above partial bundles, once taken when entering dm, the other time taken when leaving dm $\alpha_\nu I_\nu dm d\nu$ is the difference of these two sums Now, energy is obtained by directional integration of the strength Integration over all directions r of the above loss of strength shows thus that the whole $d\nu$-radiation travelling through dm per unit time loses the amount

$$dm d\nu \alpha_\nu(P) \int I_\nu(P, r)\, d\omega \qquad \ldots(12)$$

in consequence of absorption

In the same way it is seen that the amount

$$dm d\nu \sigma_\nu(P) \int I_\nu(P, r)\, d\omega \qquad \ldots\ldots(13)$$

is scattered within dm. We now show how this scattered radiation is redistributed among the different directions r'. According to the above considerations, the radiation flowing through dm per unit time and spreading out within a solid angle $d\omega$ (containing r) loses the amount

$$E = dm d\nu \sigma_\nu(P)\, I_\nu(P, r)$$

through scattering within dm. This scattered amount E will be distributed over the other directions r' according to the law of

scattering. The part falling within a cone $d\omega'$ about the direction r' is therefore

$$\frac{E}{4\pi}\gamma_\nu(P,r,r')\,d\omega'$$

Summation over all cones $d\omega$, i.e. integration through all incident directions r, gives, therefore,

$$dm\,d\omega'\,d\nu\,\frac{\sigma_\nu(P)}{4\pi}\int I_\nu(P,r)\,\gamma_\nu(P,r,r')\,d\omega \qquad \text{....(14)}$$

as the part of the scattered radiation (13) which is redistributed within the cone $d\omega'$ If we now integrate this over all directions r', we obtain, according to (11), again (13) as it should be.

We note for later purposes that the parallel bundle of scattered radiation emitted by dm into the direction r has the strength (the notation for incident and scattered ray is interchanged)

$$dm\,d\nu\,\frac{\sigma_\nu(P)}{4\pi}\int I_\nu(P,r')\,\gamma_\nu(P,r',r)\,d\omega'. \qquad \text{......(14')}$$

Equation of transfer Let us consider a light ray of direction r through P, and let P' be a nearby point lying in the direction r from P, $ds=\overline{PP'}$ We construct a thin cylinder about ds with the two bases at P and P', and with the cross-section dq. The strength at P',

$$I_\nu(P',r)\,dq\,d\nu, \qquad \text{......(15)}$$

of the parallel bundle of direction r through dq consists then of three different parts Firstly, the strength at P of the same bundle weakened by absorption and scattering in the cylinder,

$$\{1-\rho(\alpha_\nu+\sigma_\nu)\,ds\}\,I_\nu(P,r)\,dq\,d\nu,$$

secondly, the strength (8) of the parallel bundle emitted by the cylinder in direction r, with

$$dm=\rho\,dq\,ds,$$

thirdly, the strength (14') of the bundle of scattered radiation sent by the cylinder into the direction r On dividing the equation obtained in this way through $\rho\,dq\,ds\,d\nu$, we get Schwarzschild's fundamental equation of transfer

$$\frac{1}{\rho(\alpha_\nu+\sigma_\nu)}\,\frac{dI_\nu(P,r)}{ds}=J_\nu(P,r)-I_\nu(P,r), \qquad \text{......(16)}$$

together with

$$(\alpha_\nu+\sigma_\nu)\,J_\nu(P,r)=\eta_\nu(P)+\frac{\sigma_\nu(P)}{4\pi}\int I_\nu(P,r')\,\gamma_\nu(P,r',r)\,d\omega' \qquad \ldots\ (16')$$

The derivative on the left-hand side of (16) is the directional derivative with respect to P, taken in the direction r of the ray. On introducing three orthogonal axes x, y, z, $P=(x, y, z)$, and on setting

$$\alpha=\cos(r,x),\quad \beta=\cos(r,y),\quad \gamma=\cos(r,z) \qquad \ldots(16'')$$

for the direction cosines of r, we obtain

$$\frac{dI}{ds}=\alpha\frac{\partial I}{\partial x}+\beta\frac{\partial I}{\partial y}+\gamma\frac{\partial I}{\partial z} \qquad \ldots(16''')$$

The quantity $J_\nu(P,r)$ introduced by (16′) may, according to Schwarzschild, be called the *Ergiebigkeit* It is an average of the two quantities

$$\frac{\eta_\nu}{\alpha_\nu},\quad \frac{1}{4\pi}\int I_\nu\gamma_\nu\,d\omega',$$

having thus the dimension of an intensity. The Ergiebigkeit depends, in general, upon the direction r. In the case of uniform scattering ($\gamma_\nu=1$) or of isotropic radiation (the intensity is independent of r), however, it is a function of P only

The equation of transfer is of great generality It does not require that radiation is the only mode of energy transfer, it just refers to the part of energy appearing in the form of radiation

§ 3 RADIATIVE EQUILIBRIUM. RADIATION AND TEMPERATURE

If $4\pi\epsilon(P)\,dm$ denote the heat *energy liberated* in dm, i.e. the sum, net gain within dm of heat energy due to convection and conduction plus the radiant energy (of all frequencies) coming from sub-atomic sources within dm per unit time, the conservation of energy is expressed by the equation

$$\int_S \mathfrak{F}_n\,d\sigma=4\pi\int_V \rho\epsilon\,dv, \qquad \ldots\ldots(17)$$

where V is an arbitrary volume, S its surface and n the outward

normal on S. According to (5), integrated through the whole spectrum, this is equivalent to the equation

$$\operatorname{div} \mathfrak{F} = 4\pi\rho\epsilon \qquad \ldots\ldots(17')$$

It is of importance to realize that (17) or (17′) refers only to the total radiation of all frequencies. Scattering has no influence upon the wave-length, the absorbed radiation is, however, re-emitted within other parts of the spectrum, implying, in general, an exchange of energy between the different parts of the spectrum.

Integration through all directions r of the equation of transfer (after multiplication with $\rho(\alpha_\nu + \sigma_\nu)$) yields, according to (11), the equation

$$\operatorname{div} \mathfrak{F}_\nu = 4\pi\rho\eta_\nu - \rho\alpha_\nu \int I_\nu \, d\omega, \qquad \ldots\ldots(18)$$

the scattered radiation having dropped out as it should do. On integrating (18) through the whole spectrum we obtain, according to (17′), the fundamental relation

$$\eta(P) = \frac{1}{4\pi}\int d\omega \int_0^\infty \alpha_\nu(P)\, I_\nu(P, r)\, d\nu + \epsilon(P). \qquad \ldots(19)$$

If (19) is multiplied by $4\pi dm$, the left-hand side becomes, according to (8), the total emission of dm, whilst the first term on the right becomes the part of the total radiation that is absorbed by dm. (19) means therefore that the radiation emitted minus the radiation absorbed equals the energy liberated.

Since (18) is a consequence of the equation of transfer only, (17′) follows, conversely, from (19). (17′) and (19) are therefore equivalent expressions of the conservation of energy.

In the case $\epsilon = 0$, used as a first approximation in the outer layers of a star, we speak of strict *radiative equilibrium*. Energy is transferred by radiation only. It should be noted that ϵ could be negative as well as positive in the general case. When, for instance, the element loses heat by convection or conduction, subatomic energy being missing, ϵ is negative.*

* (17′), (19) possess, of course, general validity, as they express the energy principle. They are, however, of importance only when radiation is the principal agent of energy transfer. It is in this sense that (19) is spoken of as the equation of radiative equilibrium.

Pure absorption Gray material We consider the case $\sigma_\nu \equiv 0$. Furthermore, let us suppose that the absorption coefficient is independent of the wave-length, $\alpha_\nu = \alpha$ (gray material). This case is of special importance because of its simplicity The fundamental equations (16), (16′) and (19) become, after integration through the spectrum,

$$\frac{1}{\sigma(P)} \frac{dI(P, r)}{ds} = J(P) - I(P), \quad . \quad (20)$$

$$J(P) = \frac{1}{4\pi} \int I(P, r)\, d\omega + \frac{\epsilon(P)}{\sigma(P)} \quad (20')$$

Monochromatic radiative equilibrium In the case of pure scattering, $\sigma_\nu = \eta_\nu = 0$, we know that the wave-length remains unchanged The scattered radiation is fully re-emitted with the same frequency and distributed through the different directions This state of affairs is often called monochromatic radiative equilibrium The equations of transfer (16), (16′) take, here, the form

$$\frac{1}{\sigma_\nu(P)} \frac{dI_\nu(P, r)}{ds} = J_\nu(P, r) - I_\nu(P), \quad . \quad .(21)$$

$$J_\nu(P, r) = \frac{1}{4\pi} \int I_\nu(P, r')\gamma_\nu(P, r', r)\, d\omega' \quad \ldots \ldots (21')$$

In this case, we have, of course, according to (18),

$$\operatorname{div} \mathfrak{F}_\nu = 0. \quad \ldots (22)$$

It is physically plain that the case of purely absorbing gray material in strict radiative equilibrium is formally equivalent to the case of monochromatic radiative equilibrium and uniform scattering, $\gamma_\nu \equiv 1$. The absorbed radiation is uniformly re-emitted in direction, in the same way as the scattered radiation is uniformly redistributed in direction The equations (20), (20′), $\epsilon = 0$, have accordingly the same form as (21), (21′), $\gamma_\nu = 1$. This formal equivalence is of importance, since it allows us to develop the same mathematical theory for the two physically different cases.

Local thermodynamical equilibrium. In the case of pure scattering, radiation has no relation to the temperature. As soon,

however, as absorption and emission play a rôle, we should have information how the radiation is related to the temperature of the matter

Let $B_\nu = B_\nu(T)$ be the intensity of the ν-radiation of a black body at temperature T,

$$B_\nu = \frac{2h\nu^3/c^3}{e^{h\nu/kT}-1}, \tag{23}$$

where h signifies Planck's constant, k Boltzmann's constant and c the velocity of light For the total radiation we have

$$B = \int_0^\infty B_\nu d\nu = \frac{\sigma}{\pi} T^4, \tag{24}$$

σ being Stefan's constant.

When the radiation is enclosed between black walls, it obeys Kirchhoff's law

$$\eta_\nu = \alpha_\nu B_\nu \tag{25}$$

The condition of being enclosed is, of course, not rigorously fulfilled in the stars, since there is always a radial net flux of radiation Eddington's perfect gas star, however, has such a high opacity that the radiation is practically enclosed, the degree of accuracy being higher than in laboratory experiments Astrophysics uses the term 'local thermodynamical equilibrium' in order to express that the radiation behaves like an enclosed one, i.e that Kirchhoff's law holds Even in the outer layers of a star this state of affairs has proved to be a useful approximation

§ 4 THE MAIN PROBLEM

In the physics of the outside of a star, the following problem plays an essential rôle. The radiation field, in particular the radiation emergent from the surface and its distribution in direction and wave-length, and the temperature distribution are to be determined when the coefficients of absorption and of scattering as well as the law of scattering and ϵ are given

This problem naturally represents only a part of the general problem of the determination of the whole physical state. The quantities regarded as given in the above problem are not strictly known in the stars Actually they enter other more or less known

physical laws (containing of course the conditions of mechanical equilibrium) which, together with the laws of radiation, should enable us to determine the total physical state The excessive difficulties, however, connected with this problem as it stands compel us to select partial problems, of which the one formulated above is one of the most important. Comparison with observational data of the solution of this problem, taken for different choices of the given quantities, has led to valuable insight into the structure of stellar atmospheres The following comparatively simple special cases have hitherto found the chief attention of astrophysicists Firstly, the Schuster-Schwarzschild model of a purely scattering atmosphere Schwarzschild was the first to approach this model with rigorous mathematical methods Secondly, Schwarzschild's (formally equivalent) model of a purely absorbing gray atmosphere ($\epsilon = 0$) The important limit case of infinite optical depth has (by approximate methods) been extensively treated by Milne, who also studied the spectral distribution of the emergent light The explanation of the law of darkening on the sun's disk is one of the main successes due to the study of those models of radiative equilibrium. Schwarzschild's investigations have, furthermore, led to the result that the origin of the Fraunhofer lines in the solar spectrum is due to scattering rather than absorption.

So far as concerns the general solution of the above problem, the following remarks are of importance. The equation of transfer (16) can be considered as a first order differential equation for the intensity. On integrating it along a ray, with respect to the boundary conditions, we obtain the intensity expressed in terms of the Ergiebigkeit If this is inserted into (16′), a linear integral equation (due to Schwarzschild) is obtained for the determination of the Ergiebigkeit In the case of pure scattering, $\sigma_\nu = \eta_\nu = 0$, this equation completely determines the Ergiebigkeit In the photospheric layers of a star, however, absorption processes are predominant, and the unknown emission $\eta_\nu = \sigma_\nu B_\nu$ enters The solution of Schwarzschild's integral equation determines, in this case, the Ergiebigkeit, and therefore the intensity I_ν as a 'func-

tional' of B_ν, for every frequency ν.* Since $B_\nu = B_\nu(B)$ is a definite function of B, the intensity appears thus a definite functional of $B = B(P)$, i.e. of the temperature distribution. If the intensity thus determined is inserted in the equation of radiative equilibrium (19), the right-hand side becomes a definite functional of $B(P)$. The left-hand side

$$\eta(P) = \int_0^\infty \alpha_\nu(P)\, B_\nu\{B(P)\}\, d\nu,$$

however, represents a definite function of the point P and of $B = B(P)$ We thus arrive at a definite functional—more precisely, at a non-linear integral equation for the determination of $B(P)$, i e of the distribution of temperature Once $B(P)$ is known, we are able to find η_ν and therefore I_ν for every frequency

It is difficult to solve the integral equation in its full generality, and the limitation to simpler special cases seems necessary. If the material is gray, i e. if α_ν is independent of ν, the equation of radiative equilibrium simplifies considerably If, however, α_ν varies with ν, the integral equation still remains non-linear. In gray material without scattering we have the simplest case, since the integral equation then becomes linear, and the determination of $B(P)$ appears entirely separated from the determination of the spectral distribution, the latter being a matter of direct integration. It should be noted, that the mathematically more general case, where α_ν and σ_ν are both independent of ν, leads to the same formal integral equation (linear) as in case $\sigma_\nu = 0$. But the spectral distribution of light must then be determined by the solution of Schwarzschild's integral equation.

The greater part of this tract will be devoted to the mathematical problems arising from the two particular cases, pure scattering, and pure absorption in gray material The first case plays, as mentioned before, a chief rôle in the outermost layers of a star, whilst the second case has found chief application to the photospheric layers.

* This determination of J_ν is possible by means of the Neumann series for the solution of the integral equation. Convergence and uniqueness of that solution are treated in § 34.

So far as concerns the boundary conditions, the radiation incident at the surface of an isolated star is zero, while the radiation at great depth becomes isotropic In the case of a close binary star, the radiation of the other component must be taken account of Or, in the case of a 'planetary atmosphere' in radiative equilibrium, the solar radiation must be considered. In all these cases the radiation incident at the surface is given Another type of boundary condition occurs in Milne's spherical model of a 'planetary nebula' in radiative equilibrium. The inner face of the shell receives not only the radiation of the central star but also the light coming from other parts of the inner face The boundary condition appears here as a relation between incident and emergent radiation.

§ 5 SCHWARZSCHILD'S MODELS MATERIAL STRATIFIED IN PARALLEL PLANES

In the atmospheres of celestial bodies the material may be considered stratified in concentric spheres The radiation field having spherical symmetry too, the intensity $I(P, r)$ becomes a function of the distance a from the centre and of the angle θ between the radius vector and the direction r only, $I = I(a, \theta)$, $0 \leqq \theta \leqq \pi$ In this case we obtain

$$\frac{dI}{ds} = \cos\theta \frac{\partial I}{\partial a} - \frac{\sin\theta}{a} \frac{\partial I}{\partial \theta}.$$

When, instead of a, θ, the variables

$$\xi = a\cos\theta, \quad \eta = a\sin\theta$$

are introduced, we have more simply

$$\frac{dI}{ds} = \frac{\partial I}{\partial \xi}$$

Neglection of curvature. Let us consider a layer $a_1 \geqq a \geqq a_0$ of the star. If, the depth $a_1 - a_0$ being kept fixed, the radius a_1 tends to infinity, we formally obtain the limit case of a plane slab Another way of getting this limit case is the following. We set

$$x = x_0 \frac{a_1 - a}{a_1 - a_0}, \quad \rho(\sigma_\nu + \sigma_\nu) = \frac{x_0}{a_1 - a_0} f(x),$$

x_0 being a fixed quantity, while $f(x)$ is a fixed function of x,

$0 < x < x_0$. On proceeding to the limit $a_0 \rightarrow a_1$, i e. on making the shell thinner and thinner and making $\rho(\alpha_\nu + \sigma_\nu)$ larger and larger in the same ratio (at homologous points), we obtain, as seen from (16) and from the above expression of dI/ds, again the limit case of a plane slab. This time the radius has been kept constant. It is customary to neglect the curvature in the outer layers of a star, the material being thus stratified in parallel planes Let x generally denote the depth of the point P below a fixed layer, x being reckoned negative for points above that plane The direction r is characterized by two angles θ, ϕ (we retain the second angle ϕ in view of later applications), θ being the angle between r and the direction of negative x (outward normal of the slab), $0 \leqq \theta \leqq \pi$ ϕ is the azimuth, i e the angle between the plane, containing r and the x-axis, and a fixed plane through that axis, $0 \leqq \phi \leqq 2\pi$. We shall only consider the case where the radiation field is the same in all points with the same x, I_ν becoming then a function of (x, r) alone. We have

$$ds = -\sec\theta\, dx, \qquad \ldots..(26)$$

thus yielding

$$\frac{dI}{ds} = -\cos\theta \frac{\partial I}{\partial x}.$$

The Schwarzschild-Milne model. This is the classical case of purely absorbing and gray material in local thermodynamical equilibrium. It is convenient to introduce, instead of x, the *optical depth*

$$\tau = \int_{-\infty}^{x} \sigma\rho\, dx$$

below the surface In the case of strict radiative equilibrium, the fundamental equations (20), (20′) take then, according to (25) and (26), the remarkably simple form

$$\cos\theta \frac{\partial I(\tau, r)}{\partial\tau} = I(\tau, \theta) - J(\tau) \qquad \ldots..(27)$$

and

$$J(\tau) = \frac{1}{4\pi}\int I(\tau, r)\, d\omega, \qquad \ldots.(28)$$

where, in the case of local thermodynamical equilibrium,

$$J = \frac{\eta}{\sigma} = \int_0^\infty B_\nu\, d\nu = B$$

If convection, conduction or subatomic energy sources are not negligible, we must add the term ϵ/α on the right-hand side of (28). An important fact is that the absorption coefficient appears only indirectly, namely through the mediation of the new independent variable τ It is to be remarked that (27), (28) determine the total radiation field, in particular the direction distribution of the emergent light, without any hypothesis about temperature and its relation to the radiation We need it, however, as soon as the spectral distribution, or the temperature distribution, is wanted

Let us introduce the notation

$$\int_{+} = \int_{\theta<\pi/2}, \quad \int_{-} = \int_{\theta>\pi/2}$$

for the integrals over the hemispheres of all outward and all inward directions We set

$$\pi F_{+}(\tau) = \int_{+} I(\tau, \imath)\cos\theta\, d\omega, \quad \pi F_{-}(\tau) = \int_{-} I(\tau, \imath)\,|\cos\theta|\, d\omega. \quad \ldots.(29)$$

The equations

$$\int_{+} \cos\theta\, d\omega = \int_{-} |\cos\theta|\, d\omega = \pi,$$

following from

$$d\omega = \sin\theta\, d\theta\, d\phi,$$

show that F_{+} and F_{-} represent averages of the intensity within the respective hemispheres. Furthermore, we set

$$\pi F = \pi F_{+} - \pi F_{-} = \int I \cos\theta\, d\omega \quad \ldots(30)$$

The flux vector $\mathfrak{F}$ is, of course, normal to the layers, and we have, according to (17), $\epsilon = 0$,

$$\mathfrak{F} = \pi F = \text{const.} \quad \ldots.. (31)$$

Another important relation, due to Eddington, is obtained if (27) is multiplied by $\cos\theta$ and integrated through all directions According to (30) and (31) we get

$$K \equiv \frac{1}{\pi}\int I \cos^2\theta\, d\omega = F\tau + \text{const.} \quad \ldots(32)$$

The physical meaning of K is that it equals c/π times the radiation pressure normal to the layer.

In the absence of strict radiative equilibrium, $\epsilon \neq 0$, the fundamental equations are

$$\cos\theta \frac{\partial I}{\partial \tau} = I - J, \quad \ldots(33)$$

$$J = \frac{1}{4\pi}\int I\, d\omega + \frac{\epsilon}{\alpha}. \quad \ldots(34)$$

Instead of (31) we have

$$\frac{dF}{d\tau} = -4\frac{\epsilon}{\alpha}, \quad \ldots(35)$$

and (32) must be replaced by

$$\frac{dK}{d\tau} = F \quad \ldots(36)$$

Another simplification of the model is effected by assuming the slab to have infinite optical depth,

$$0 \leqq \tau < \infty.$$

This corresponds well to the conditions within a star (large opacity in the interior) We formulate then the first standard problem, treated approximately by Milne:

PROBLEM I $I(\tau, \imath) \geqq 0$ and $B(\tau) \geqq 0$, $\tau < \infty$ are to be determined from (27) and (28), the radiation incident at the surface, $\tau = 0$,

$$I(0, \imath) \geqq 0, \quad \theta > \pi/2,$$

being given, as well as the flux constant F.

The incident radiation is zero for ordinary stars; positive, however, for close binaries (or in Milne's model of a planetary atmosphere subject to insolation) For an ordinary star, the flux constant is derived from the observed effective temperature T_e. At the surface we have $F = F_+$, this being an average of the intensity of the emergent radiation. If this were due to a black body, the intensity would be $\sigma T^4/\pi$, T being the surface temperature In general,

$$F = \frac{\sigma}{\pi} T_e^4 \quad \ldots(37)$$

defines the effective temperature T_e.

The relations (27) and (28) being linear in I, $J = B$, the solution of Problem I will be the sum of the solutions of two partial problems

PROBLEM I*a* Problem I in the case

$$I(0, r) = 0, \quad \theta > \pi/2, \qquad \ldots (38)$$

but for an arbitrary $F \geqq 0$.

PROBLEM I*b* Problem I in the case

$$F = 0 \qquad \ldots (39)$$

In lack of strict radiative equilibrium, $\epsilon \neq 0$, we have an analogous problem We confine ourselves to the case (38)

PROBLEM II $I(\tau, r) \geqq 0$ and $B(\tau) \geqq 0$ are to be determined from (33) and (34), the incident radiation being zero, the net flux F_0 at the surface being given, and ϵ/σ being given as a function of τ.

The Schuster-Schwarzschild model This is the case of monochromatic radiative equilibrium, with $\epsilon = 0$ The affix ν can be omitted, the frequency being always the same. On introducing an analogous optical depth

$$\tau = \int_{-\infty}^{z} \rho \sigma \, dz,$$

the fundamental equations (21), (21′) take, according to (26), the form

$$\cos\theta \frac{\partial I(\tau, r)}{\partial \tau} = I(\tau, r) - J(\tau, r), \qquad \ldots (40)$$

$$J(\tau, r) = \frac{1}{4\pi} \int I(\tau, r') \gamma(\tau, r', r) \, d\omega'. \qquad \ldots (41)$$

According to (22), the monochromatic net flux is constant. It is to be noted that Eddington's relation (32) holds too For multiplication of (40) by $\cos\theta$ and integration with respect to r yields

$$\frac{dK}{d\tau} = F - \frac{1}{\pi} \int J(\tau, r) \cos\theta \, d\omega.$$

If, here, the integral is evaluated by means of (41), the integrand of the thus obtained r'-integral is seen to contain the factor

$$\int \cos\theta \, \gamma(\tau, r, r') \, d\omega = 0.$$

The vanishing is a consequence of (11″), i e. of the fact that γ is an even function on the r-sphere, $\cos\theta$ being an odd function.

PROBLEM III Case of infinite optical depth. $I(\tau,\vartheta)\geqq 0$ and $J(\tau,\vartheta)\geqq 0$ are to be determined from (40) and (41), the incident radiation being zero and the law of scattering γ as well as the flux constant F being given

PROBLEM IV Case of finite optical depth, $0<\tau<\tau^\star$ I and J are, for a given law of scattering, to be determined, when the radiation incident at the outer face is zero and when the radiation incident at the inner face,

$$I(\tau^\star,\theta), \quad \theta<\pi/2,$$

is given.

In the case of uniform scattering, $\gamma\equiv 1$, Problem III is formally the same as Problem I*a* The case $\gamma=1$ and $I(\tau^\star,\theta)=$const., $\theta<\pi/2$ (black body as a background), is the classical problem treated by Schwarzschild in a fundamental memoir. Problems III and IV have hitherto not been discussed in full generality. The existence and uniqueness of the solution will be discussed in Chapter IV Furthermore, limits for the solution will be given that are independent of the law of scattering They are the same limits which were found by Schwarzschild in the case of uniform scattering In this classical case, considerably narrower limits will moreover be found

§ 6. FIRST CONSEQUENCES. THE RADIATION FROM GREAT DEPTH

We first draw some simple conclusions from the positivity of the intensity and of the Ergiebigkeit, confining ourselves to the two standard cases formulated in the preceding section. We suppose that

$$\epsilon\geqq 0 \text{ for all sufficiently large } \tau.$$

(35) shows then that $F(\tau)$ never increases, for τ sufficiently large. From (36), we have

$$K(\tau)-K(0)=\int_0^\tau F(t)\,dt. \qquad \text{......(42)}$$

Since, according to $I\geqq 0$, K is never negative, (42) implies

$$\int_0^\tau F\,dt\geqq -K(0). \qquad \text{......(43)}$$

This proves that $F \geqq 0$ holds for all large enough τ, i.e. that the net flux is outward for these τ, for the opposite assumption would, according to the decreasing of F, imply that the left-hand side of (43) tends to $-\infty$ as $\tau \to \infty$. These considerations show also that the limit

$$F_\infty = \lim_{\tau=\infty} F(\tau) \geqq 0$$

exists. From this fact and from (42) we obtain the important relation

$$F_\infty = \lim_{\tau=\infty} \frac{1}{\pi\tau} \int \cos^2\theta \, I(\tau, \vartheta)\, d\omega. \qquad \text{.. (44)}$$

The generality of this equation must be emphasized. In strict radiative equilibrium, $\epsilon = 0$, F_∞ is of course the flux constant F. (44) holds in the Schuster-Schwarzschild model too, because Eddington's relation was found to hold in that case.

From (35) we now get

$$\pi F_\tau = \pi F_\infty + 4\pi \int_\tau^\infty \frac{\epsilon}{\alpha}\, d\tau, \quad F_\infty \geqq 0. \qquad \text{......(45)}$$

The second term, taken for $\tau = 0$,

$$4\pi \int_0^\infty \frac{\epsilon}{\alpha}\, d\tau = 4\pi \int_{-\infty}^\infty \rho\epsilon\, dx,$$

represents, as it should do, the energy liberated within a normal cylinder through the whole slab, having the unit area as its cross-section. As a special case of (45), we note that

$$F_\infty = F_0 - 4 \int_0^\infty \frac{\epsilon}{\alpha}\, d\tau. \qquad \text{......(45')}$$

It is thus seen that Problem II additively decomposes into two partial problems, the first one being Problem II in the case $F_\infty = 0$, while the second one is simply Problem Ia, $F = F_\infty$. We therefore confine our attention to

PROBLEM IIa. Problem II in the case

$$F_0 = 4 \int_0^\infty \frac{\epsilon}{\alpha}\, dt,$$

i.e. in the case where the net flux at the surface equals the energy liberated per unit time within a cylinder of cross-section one.

The solution of the general Problem II is obtained in adding

to the solution of Problem IIa the solution of Problem Ia with an arbitrary $F = F_\infty \geqq 0$

On solving the equation of transfer (40) with respect to the intensity, we get

$$e^{-\tau' \sec\theta} I(\tau', r) = e^{-\tau \sec\theta} I(\tau, r) - \sec\theta \int_\tau^{\tau'} e^{-t \sec\theta} J(t, r)\, dt \quad .. \quad (46)$$

It may seem surprising that no boundary condition for $\tau \to \infty$ appears in the formulation of Problems I, II and III The supposition, however, that I and J be non-negative quantities, allows us to dispense with it

THEOREM I. We have

$$I(\tau, r) = \sec\theta \int_\tau^\infty e^{-(t-\tau)\sec\theta} J(t, r)\, dt, \quad 0 \leqq \theta < \pi/2, \quad .. \quad (47)$$

i e the radiation coming from the deeper layers $(\theta < \pi/2)$ is solely due to the Ergiebigkeit of the material within the slab.

Proof Considering in (46) the case $\theta < \pi/2$, $\tau < \tau'$, we infer, according to $J \geqq 0$, that, for any fixed direction r, $\theta < \pi/2$,

$$e^{-\tau' \sec\theta} I(\tau', r), \quad 0 \leqq \theta < \pi/2,$$

decreases with increasing τ'. According to $I \geqq 0$ this implies the existence of the limit

$$i(r) = \lim_{\tau' = \infty} e^{-\tau' \sec\theta} I(\tau', r) \geqq 0,$$

for $\theta < \pi/2$ We are thus allowed to proceed to the limit $\tau' \to \infty$ in (46),

$$I(\tau, r) = e^{\tau \sec\theta} i(r) + \sec\theta \int_\tau^\infty e^{-(t-\tau)\sec\theta} J(t, r)\, d\omega,$$

for $\theta < \pi/2$. The first (non-negative) term on the right-hand side represents the radiation not due to the Ergiebigkeit of the slab. Such radiation cannot, however, exist in the steady state. We have from (48), according to $\sec\theta \geqq 1$,

$$I(\tau, r) \geqq e^\tau i(r) \geqq 0,$$

thus yielding

$$\int I(\tau, r) \cos^2\theta\, d\omega \geqq \int_+ I \cos^2\theta\, d\omega \geqq e^\tau \int_+ i \cos^2\theta\, d\omega.$$

This is, according to the finiteness of F_∞, compatible with (44) only if

$$\int_+ i \cos^2 \theta \, d\omega = 0,$$

i e if $i(r)$ vanishes for almost all directions r (in the sense of the Lebesgue measure) Values different from zero in a set of measure zero are, however, physically meaningless It should be noted that we cannot measure intensities, but only amounts of energy. Such an amount is always represented by an integral $\int I \, d\omega$ extended through a finite cone of directions The values of I in a set of directions of measure zero are here, of course, without influence

§ 7 THE INTEGRAL EQUATIONS OF THE PROBLEMS

The reduction to integral equations of boundary value problems of the theory of radiation goes back to Hilbert and Schwarzschild

Milne's integral equations of Problems I, II On setting $\tau = 0$ in (46) and writing τ instead of τ' afterwards we find, for $\theta > \pi/2$, the radiation coming from the upper layers,

$$I(\tau, r) = e^{-\tau |\sec\theta|} I(0, r) + |\sec\theta| \int_0^\tau e^{-(\tau - t)|\sec\theta|} J(t) \, dt. \qquad \dots .(48)$$

When (48) and (47), $J = J(\tau)$, are inserted into (34), a linear integral equation for $J(\tau)$ is obtained In order to find its kernel we introduce, according to Schwarzschild, the functions

$$E_n(x) = \frac{1}{2\pi} \int_+ e^{-x \sec\theta} \sec^{2-n} \theta \, d\omega \qquad \dots ..(49)$$

$$= \frac{1}{2\pi} \int_- e^{-x|\sec\theta|} |\sec^{2-n} \theta| \, d\omega.$$

On setting $s = |\sec\theta|$ they appear, according to $d\omega = \sin\theta \, d\theta \, d\phi$, in the well-known form

$$E_n(x) = \int_1^\infty e^{-sx} s^{-n} \, ds. \qquad \dots\dots(50)$$

Insertion of (47) and (48) into (34) yields two essential terms, the first one being

$$\frac{\epsilon}{\sigma} + \frac{1}{4\pi} \int_- e^{-\tau |\sec\theta|} I(0, r) \, d\omega,$$

due to the energy liberated and to the incident radiation, while the second term is the sum of two double integrals

$$\frac{1}{4\pi}\int_{+}\int_{\tau}^{\infty} \sec\theta\, e^{-(t-\tau)\sec\theta} J(t)\,dt\,d\omega$$
$$+\frac{1}{4\pi}\int_{-}\int_{0}^{\tau} |\sec\theta|\, e^{-(\tau-t)|\sec\theta|} J(t)\,dt\,d\omega. \qquad \ldots. \ (51)$$

The integrand being measurable and non-negative, we may interchange the order of integration, obtaining thus, according to (49), $n=1$,

$$\Lambda(J)_\tau \equiv \frac{1}{2}\int_0^\infty J(t)\,E(|\tau-t|)\,dt, \qquad . . (52)$$

for (51) The use of a brief symbol Λ for the linear integral operator is convenient for the treatment of the problem. We thus obtain Milne's first integral equation for the determination of the Ergiebigkeit $J(\tau)=B(\tau)$,

$$J(\tau)=\Lambda(J)_\tau+\frac{\epsilon}{\alpha}+\frac{1}{4\pi}\int_{-} e^{-\tau|\sec\theta|} I(0,r)\,d\omega. \quad \ldots(53)$$

A formal disadvantage of (53) is that it does not contain the net flux $\pi F(\tau)$ An equation containing F is, according to Milne, obtained by inserting (47) and (48) directly into (30) According to (49), $n=2$, we thus get Milne's second integral equation

$$F(\tau)=\frac{1}{\pi}\int_{-} \cos\theta\, e^{-\tau|\sec\theta|} I(0,r)\,d\omega$$
$$+2\int_\tau^\infty J(t)\,E_2(t-\tau)\,dt-2\int_0^\tau J(t)\,E_2(\tau-t)\,dt, \qquad \ldots\ldots(54)$$

containing all the data given in Problems I, II (54) goes, of course, together with (45)

If (47), (48) are inserted into the integral (32) representing K, we obtain, according to (49), $n=3$, the equation

$$K=\frac{1}{\pi}\int_{-} \cos^2\theta\, e^{-\tau|\sec\theta|} I(0,r)\,d\omega+2\int_0^\infty J(t)\,E_3(|\tau-t|)\,dt. \qquad . . .(55)$$

Since the first term on the right-hand side is not greater than

$$K_0 = \frac{1}{\pi}\int_{-} \cos^2\theta I(0, \imath)\, d\omega,$$

it follows that (44) can also be written in the form

$$F_\infty = \lim_{\tau=\infty} \frac{2}{\tau}\int_0^\infty J(t)\, E_3(|\tau - t|)\, dt \qquad .. \qquad (56)$$

For the actual solution of Problems I, II, it is more convenient to start from the integral equation (53) together with (56) and (45′).

THEOREM II Any non-negative solution $B(\tau)$ of (53), that satisfies (56) where F_∞ is determined by (45′), yields together with (47), (48) a solution of Problem II, with given energy liberated, given incident radiation and a given surface flux $\pi F_0 \geqq 4\pi\int \rho\epsilon\, da$ (integrated through the slab)

COROLLARY 1. In case of Problem I*a* we must, in (53), set $\epsilon = 0$ and $I(0, \imath) = 0$, $\theta > \pi/2$

COROLLARY 2. In case of Problem I*b* we must set $\epsilon = 0$ in (53) and $F = F_\infty = 0$ in (56).

COROLLARY 3 In case of Problem II*a* we must set $I(0, \imath) = 0$, $\theta > \pi/2$, and $F_\infty = 0$ in (56).

The proof of the theorem is plain from the above.

The integral equation for Problem III. Since the incident radiation is supposed to be zero we have from (46)

$$I(\tau, r) = |\sec\theta| \int_0^\tau e^{-(\tau - t)|\sec\theta|} J(t, r)\, dt; \quad \theta > \pi/2 \qquad \ldots \qquad (57)$$

for the radiation coming from the upper layers. From (11′), (11″) and from (41) we infer that the Ergiebigkeit is always an even function of direction, $J(\tau, -r) = J(\tau, \imath)$. Let us introduce the linear integral operator

$$\Lambda(J)_{\tau, r} = \frac{1}{8\pi}\int\int_0^\infty H(\tau, r, t, \imath')\, J(t, r')\, dt\, d\omega', \qquad \ldots\ldots(58)$$

where H is defined by

$$H(\tau, r; t, r') = |\sec\theta'|\, e^{-|\tau - t||\sec\theta'|}\gamma(\tau; r', \imath), \quad \ldots\ ..(59)$$

for any two directions r, r'. $H/8\pi$ is thus the kernel of the operator Λ. In the simplest case of uniform scattering, $\gamma = 1$, this operator reduces to the simpler Milne-operator, J being independent of direction. For an arbitrary law of scattering, however, no simplification is possible. The meaning of the general operator is, that, on inserting (47) and (57) into (41), the linear integral equation

$$J(\tau, r) = \Lambda(J)_{\tau, r} \qquad \text{.....}(60)$$

is obtained for the determination of the Ergiebigkeit. In analogy to Theorem II we have in the present case

THEOREM III. Any non-negative solution of (61) satisfying (44), $F_\infty = F$, yields together with (47) and (57) a solution of Problem III.

The integral equation for Problem IV. Since the incident radiation vanishes, (57) gives the intensity for $\theta > \pi/2$. The intensity of the radiation from the deeper layers is, however, given by

$$I(\tau, r) = e^{-(\tau^\star - \tau)\sec\theta} I(\tau^\star, r) + \sec\theta \int_\tau^{\tau^\star} e^{-(t-\tau)\sec\theta} J(t, r)\,dt,$$
$$\theta < \pi/2 \text{ ......}(61)$$

We define, in this case, the integral operator

$$L(J)_{\tau, r} \qquad \text{......}(62)$$

by replacing, in the integral in (58), the upper limit ∞ of integration by $\tau^\star$. On inserting (57) and (61) into (41), we obtain the linear integral equation

$$J(\tau, r) = L(J)_{\tau, r} + \frac{1}{4\pi}\int_+ e^{-(\tau^\star - \tau)\sec\theta'} \gamma I(\tau^\star, r')\,d\omega'$$
$$\text{......}(63)$$

for the determination of the Ergiebigkeit, $0 \leqq \tau \leqq \tau^\star$.

Schwarzschild's integral equation for uniform scattering. Since, in this case, J is independent of direction, the operator $L(J)_{\tau, r}$ simplifies to

$$L = L(J)_\tau = \frac{1}{2}\int_0^{\tau^\star} J(t)\,E_1(|\tau - t|)\,dt, \qquad \text{......}(64)$$

and the integral equation (63) becomes

$$J(\tau) = L(J)_\tau + \frac{1}{4\pi}\int_+ e^{-(\tau^\star - \tau)\sec\theta'} I(\tau^\star, \theta')\,d\omega'.$$
$$\text{......}(64')$$

The classical case treated by Schwarzschild is

$$I(\tau^\star, r) = I^\star = \text{const}, \quad \theta < \pi/2. \qquad \text{.....}(65)$$

In this case the second term in (64′) becomes, according to (49), $n=2$, simply

$$\tfrac{1}{2} I^\star E_2(\tau^\star - \tau) \qquad . \qquad (66)$$

For later purposes we add the remark that in the case

$$I(\tau^\star, r) = \cos\theta, \quad \theta < \pi/2 \qquad (67)$$

the second term in (64′) becomes, according to (49), $n=3$,

$$\tfrac{1}{2} E_3(\tau^\star - \tau) \qquad \text{...}(68)$$

Positivity of the kernel An obvious as well as important property of all the integral operators introduced above is that their kernels take only positive values We therefore call Λ, L *positive operators*. This positivity occurs in all boundary value problems of the theory of radiation An evident consequence of that property is the

LEMMA For any function $\Phi(t, r') \geqq 0$ we have $\Lambda(\Phi)_{\tau,r} > 0$ *everywhere*, unless Φ vanishes identically. The vanishing is, strictly, to be understood in the sense of the theory of the Lebesgue measure, i e. vanishing up to an inessential (τ, r)-set of measure zero.

We may express the lemma also in the following form $\Phi \geqq 0$ and $\Phi \not\equiv 0$ implies $\Lambda(\Phi)_{\tau,r} > 0$ for *all* τ, r. Another equivalent formulation is that

$$\Psi \geqq \Phi, \quad \Psi \not\equiv \Phi$$

implies the strict inequality

$$\Lambda(\Psi)_{\tau,r} > \Lambda(\Phi)_{\tau,r} \qquad \text{......}(69)$$

for *all* τ, r. The equivalence follows from the linearity of the operator, in particular from

$$\Lambda(\Psi) = \Lambda(\Phi) + \Lambda(\Psi - \Phi).$$

The same thing is, of course, true for the operator L.

§ 8 SOME PROPERTIES OF THE FUNCTIONS $E_n(x)$

For the discussion of the simpler integral equations we need some simple properties of the functions $E_n(x)$, $x \geqq 0$, $n \geqq 0$ We have $E_0(x) = e^{-x}/x$, while $E_1(x)$ becomes logarithmically infinite as $x \to 0$ For $n > 1$ all functions $E_n(x)$ are, however, continuous at $x = 0$ By partial integration in (50), the well-known recursion formula

$$nE_{n+1}(x) = e^{-x} - xE_n(x) \qquad \dots(70)$$

is obtained. We have, furthermore,

$$E_n(x) = \int_x^\infty E_{n+1}(v)\,dv. \qquad \dots(71)$$

From (50) the inequality

$$E_{n+1}(x) < E_n(x) \qquad \dots(72)$$

is obtained. Let us compare the right-hand side of (70) with the right-hand side of the equation obtained from (70) by replacing n with $n-1$ We then find, according to (72),

$$(n-1)\,E_n(x) \leqq nE_{n+1}(x) \qquad \dots(73)$$

for $n > 1$, the equality sign holding only at $x = 0$ We note that

$$E_n(0) = \frac{1}{n-1}. \qquad \dots(74)$$

If the left-hand side of (70) is combined once with (72), another time with (73), we find

$$\frac{e^{-x}}{x+n} < E_n(x) \leqq \frac{e^{-x}}{x+n-1} \qquad \dots(75)$$

for $n \geqq 1$ On applying Schwartz's inequality to

$$E_n(x) = \int_1^\infty \left(e^{-\frac{s}{2}x} s^{-\frac{n-1}{2}}\right)\left(e^{-\frac{s}{2}x} s^{-\frac{n+1}{2}}\right) ds,$$

another inequality

$$E_n^2(x) < E_{n-1}(x)\,E_{n+1}(x) \qquad \dots(76)$$

is obtained From (71) and (76) we get

$$\frac{d}{dx}\left(\frac{E_{n+1}}{E_n}\right) > 0. \qquad \dots(77)$$

For later purposes the formula

$$\frac{d}{dx}\,\frac{E_n(x-a)}{E_n(x)} < 0, \quad x > a > 0 \qquad \ldots\ldots(78)$$

will be useful. For the proof we note that the left-hand side equals, according to (71),

$$\frac{E_n(x-a)}{E_n(x)}\left\{\frac{E_{n-1}(x)}{E_n(x)} - \frac{E_{n-1}(x-a)}{E_n(x-a)}\right\}$$

By means of (77), the quantity within the parenthesis is seen to be negative

Some simple integral formulae will, furthermore, be needed From (71) and (74), we have

$$\int_0^\tau E_n(\tau-t)\,dt = \frac{1}{n} - E_{n+1}(\tau), \quad \int_\tau^{\tau^\star} E_n(t-\tau) = \frac{1}{n} - E_{n+1}(\tau^\star - \tau) \qquad \ldots\ldots(79)$$

Partial integration yields, according to (71),

$$\int_0^\tau tE_n(t)\,dt = \frac{1}{n+1} - E_{n+2}(\tau) - \tau E_{n+1}(\tau) \qquad \ldots\ldots(80)$$

On combining (79) and (80) we get

$$\int_0^\tau tE_n(\tau-t)\,dt = \int_0^\tau (\tau-t)\,E_n(t)\,dt = E_{n+2}(\tau) - \frac{1}{n+1} + \frac{\tau}{n} \qquad \ldots\ldots(81)$$

and

$$\begin{aligned}\int_\tau^{\tau^\star} tE_n(t-\tau)\,dt &= \int_0^{\tau^\star-\tau} (t+\tau)\,E_n(t)\,dt \\ &= \frac{1}{n+1} + \frac{\tau}{n} - E_{n+2}(\tau^\star-\tau) - \tau^\star E_{n+1}(\tau^\star-\tau).\end{aligned} \qquad \ldots\ldots(82)$$

CHAPTER II

SOLUTION OF PROBLEMS I AND II

§ 9 THE SOLUTION OF PROBLEM Ia

Milne's integral equation of the Problem Ia is, according to (38) and (53), $\epsilon=0$ and $J=B$,

$$B(\tau)=\Lambda(B)_\tau \qquad \ldots\ldots(83)$$

Physical reasoning would lead us to the conjecture that, in strict radiative equilibrium, the radiation becomes nearly isotropic in great depth, i e according to (28), that $I(\tau, \imath)/B(\tau)\to 1$ as $\tau\to\infty$ (44) would then show that $B(\tau)$ is asymptotically linear,

$$F=\frac{1}{\pi}\int\cos^2\theta\, d\omega \,.\lim_{\tau=\infty}\frac{B(\tau)}{\tau},$$

i.e. that $B(\tau)=\frac{3}{4}F\tau$ for large τ. We have, indeed, the

THEOREM IV. Problem Ia has a solution

$$J(\tau)=B(\tau)=\tfrac{3}{4}Ff(\tau), \qquad \ldots\ldots(84)$$

where f satisfies Milne's integral equation (83) and the inequalities

$$f(\tau)=\tau+q(\tau), \quad \tfrac{1}{2}<q(\tau)<1 \qquad \ldots\ldots(85)$$

Proof From the definition (52) of the operator Λ and from (79), $\tau^\star=\infty$, we have

$$1=\Lambda(1)_\tau+\tfrac{1}{2}E_2(\tau). \qquad \ldots\ldots(86)$$

Furthermore, from (81) and (82), $n=1$, $\tau^\star=\infty$,

$$\tau=\Lambda(t)_\tau-\tfrac{1}{2}E_3(\tau). \qquad \ldots\ldots(87)$$

On combining (86) and (87) we get

$$\tau+c=\Lambda(t+c)_\tau+\tfrac{1}{2}\{cE_2(\tau)-E_3(\tau)\} \qquad \ldots\ldots(88)$$

The smallest value of c that makes the second term $\geqq 0$ is, according to $E_2>E_3$ and $E_2\sim E_3$ for large τ, $c=1$ We thus get, for all τ,

$$\bar f(\tau)>\Lambda(\bar f)_\tau, \quad \bar f(\tau)=\tau+1. \qquad \ldots\ldots(89)$$

On the other hand, $c=\frac{1}{2}$ is, according to $2E_3\geqq E_2$, the greatest value of c making the second term $\leqq 0$ for all τ. We thus have

$$f_1(\tau)<\Lambda(f_1)_\tau, \quad f_1(\tau)=\tau+\tfrac{1}{2} \qquad \ldots\ldots(90)$$

for all $\tau>0$.

We now set successively

$$f_{n+1}(\tau)=\Lambda(f_n)_\tau \qquad \ldots.(91)$$

The inequality (90) has then the simple form

$$f_1(\tau)<f_2(\tau), \quad \tau>0 \qquad \ldots(92)$$

According to the positivity of the operator Λ, we have from (92)

$$\Lambda(f_1)_\tau<\Lambda(f_2)_\tau, \quad \tau\geqq 0,$$

or, with regard to (91), $f_2(\tau)<f_3(\tau)$, $\tau\geqq 0$ Successive application of the operation Λ generally shows that

$$f_n(\tau)<f_{n+1}(\tau), \quad \tau\geqq 0, \qquad \ldots(93)$$

i e each function of the sequence f_n lies above the preceding one

From $$f_1(\tau)<\bar{f}(\tau), \quad \tau\geqq 0,$$

we now have, again according to the positivity of Λ,

$$f_2(\tau)=\Lambda(f_1)_\tau<\Lambda(\bar{f})_\tau, \quad \tau\geqq 0$$

From (89) we get therefore $f_2(\tau)<\bar{f}(\tau)$. If the operation Λ is applied a second time, we get

$$f_3(\tau)=\Lambda(f_2)_\tau<\Lambda(\bar{f})_\tau, \quad \tau\geqq 0,$$

whence, according to (89), $f_3(\tau)<\bar{f}(\tau)$ follows. In the same way we obtain the general inequality

$$f_n(\tau)<\Lambda(\bar{f})_\tau<\bar{f}(\tau), \quad \tau\geqq 0. \qquad \ldots(94)$$

(93) and (94) evidently show that the limit function

$$f(\tau)=\lim_{n=} f_n(\tau)$$

exists and that it lies between the limits $f_1=\tau+\frac{1}{2}$ and $\bar{f}=\tau+1$, $\tau\geqq 0$. It follows from well-known theorems that, in (91), we can proceed to the limit $n=\infty$ under the integral sign. $f(\tau)$ is therefore a solution of $f=\Lambda(f)$ with the property (85). The smoothing property of an integral operator shows, finally, that $f(\tau)$ is a continuous function.

On setting $$B(\tau)=cf(\tau),$$

it remains to find the relation between c and the flux constant F From $f=\tau+q$ and from (81), (82), $n=3$, $\tau^*=\infty$, we find

$$\int_0^\infty f(t)\,E_3(|\tau-t|)\,dt=\tfrac{2}{3}\tau+E_5(\tau)+\int_0^\infty q(t)\,E_3(|\tau-t|)\,dt.$$

The last term being, according to $0<q<1$ and to (79), $n=3$, $\tau^{\star}=\infty$, less than $\frac{2}{3}$, we infer from (56), $J=cf$, $F_{\infty}=F$, that F equals $\frac{4}{3}c$, i e that (84) holds, q.e d.

From $f=\Lambda(f)$ and from (87) we infer that the remainder function $q(\tau)$ satisfies the integral equation

$$q(\tau)=\Lambda(q)_{\tau}+\tfrac{1}{2}E_3(\tau) \qquad \text{......(95)}$$

Let us, now, insert (84), $f=\tau+q$, into the second integral equation (54), where ϵ and the incident radiation are to be omitted. Making use of (81), (82), $n=2$, $\tau^{\star}=\infty$, we find that $q(\tau)$ also satisfies the integral equation

$$\int_{\tau}^{\infty} q(t)\, E_2(t-\tau)\, dt-\int_0^{\tau} q(t)\, E_2(\tau-t)\, dt=E_4(\tau) \quad \text{..(96)}$$

On inserting (84) finally into (55), we obtain by means of similar computations, and according to (32), the equation

$$\int_0^{\infty} q(t)\, E_3(|\tau-t|)\, dt=a-E_5(\tau), \qquad \text{...(97)}$$

a being a constant, the determination of which will be postponed.

From (22), (34) and (87) we find the temperature T as function of the optical depth τ,

$$\left(\frac{T_{\tau}}{T_e}\right)^4=\frac{B(\tau)}{F}=\tfrac{3}{4}f(\tau). \qquad \text{.....(98)}$$

Formulae (47), (57) for the intensity can be written in the form

$$I(\tau,r)=\int_0^{\infty} e^{-s} J(\tau+s\cos\theta, r)\, ds; \quad \theta<\pi/2, \qquad \text{......(99)}$$

$$I(\tau,r)=\int_0^{\tau|\sec\theta|} e^{-s} J(\tau+s\cos\theta, r)\, ds, \quad \theta>\pi/2. \quad \text{...(99')}$$

In the present case we have $I=I(\tau,\theta)$ and $J=J(\tau)=B(\tau)$ On inserting (84), (85) into (99) we get

$$I(\tau,\theta)=\tfrac{3}{4}F\left[\tau+\cos\theta+\int_0^{\infty} e^{-s} q(\tau+s\cos\theta)\, ds\right],$$

$$0\leqq\theta<\pi/2, \qquad \text{......(100)}$$

for the radiation coming from the deeper layers, in particular,

$$I(0,\theta)=\tfrac{3}{4}F\left[\cos\theta+\int_0^\infty e^{-s}q(s\cos\theta)\,ds\right] \quad \ldots. \quad (101)$$

for the radiation emerging at the surface

It is to be emphasized that the emergent radiation is independent of the particular behaviour of α within the slab, provided that the material is gray Since Problem III for uniform scattering formally coincides with Problem Ia, we may also say that the emergent radiation (for a given frequency) does not depend on the course of $\sigma_\nu(a)$ It should be repeated that (101) also holds independent of any assumption about temperature In the case of the sun, (101) represents the law of darkening on the solar disk, in good agreement with the observations.

§ 10 UNIQUENESS OF THE SOLUTION

The uniqueness of the solution of Problem I a follows from the

THEOREM V A non-negative solution of the integral equation (83) is necessarily of the form $B(\tau)=\text{const}\, f(\tau)$

Proof Let $B(\tau)$ be a non-negative solution of (83) We set

$$b=\text{greatest lower bound of } \frac{B(\tau)}{f(\tau)} \quad \ldots\ldots(102)$$

and $$B^\star(\tau)=B(\tau)-bf(\tau),$$

$f(\tau)$ being the solution found in the preceding section. $B^\star$ is then also a solution of (83), with the properties

$$B^\star(\tau)\geqq 0, \quad \ldots\ldots(103)$$

$$\text{Gr.l b. of } \frac{B^\star(\tau)}{f(\tau)}=0. \quad \ldots\ldots(104)$$

All we have to prove is that $B^\star(\tau)$ identically vanishes.

From (104) and from the definition of the greatest lower bound we infer the existence of an infinite sequence of numbers τ_ν, $\nu=1, 2, 3, \ldots$, such that

$$\lim_{\nu=\infty}\frac{B^\star(\tau_\nu)}{f(\tau_\nu)}=0. \quad \ldots\ldots(105)$$

Here, we must distinguish between two different cases, (*a*) the τ_ν have a finite limit point $\tau^\star$, (*b*) we have $\tau_\nu \to \infty$ as $\nu \to \infty$

Let us first treat case (*a*) It is always possible to pick out of the τ_ν a subsequence converging to $\tau^\star$ No generality is lost in assuming that $\tau_\nu \to \tau^\star$ as $\nu \to \infty$ From (105) and from $f(\tau^\star) > 0$ we have

$$\lim_{\nu=\infty} B^\star(\tau_\nu) = 0,$$

whence, according to the integral equation,

$$\lim_{\nu=\infty} \Lambda(B^\star)_{\tau_\nu} = 0$$

follows. As a function of τ, $\Lambda(B^\star)_\tau$ has the property of lower semicontinuity, the proof of which fact may be postponed,

$$\underline{\lim}_{\nu=\infty} \Lambda(B^\star)_{\tau_\nu} \geqq \Lambda(B^\star)_{\tau^\star}.$$

This shows with respect to (103) and according to the positivity of Λ, that $B^\star(\tau) \equiv 0$ The lower semicontinuity is proved in the following way. According to (103), we have, for

$$\tau^\star - \delta < \tau < \tau^\star + \delta,$$

$$\Lambda(B)_\tau \geqq \frac{1}{2}\int_0^{\tau^\star-\delta} B(t)\,E_1(\tau - t)\,dt + \frac{1}{2}\int_{\tau^\star+\delta}^{\infty} B(t)\,E_1(t-\tau)\,dt, \quad \ldots (106)$$

δ being a positive number less than $\tau^\star$. (In case $\tau^\star = 0$ the first term is to be omitted) The left-hand side is, now, a continuous function within $(\tau^\star - \delta, \tau^\star + \delta)$, the logarithmic singularity of E_1 being excluded. We have therefore

$$\underline{\lim}_{\tau\to\tau^\star} \Lambda(B)_\tau \geqq \frac{1}{2}\int_0^{\tau^\star-\delta} BE_1(\tau^\star - t)\,dt + \frac{1}{2}\int_{\tau^\star+\delta}^{\infty} BE_1(t-\tau^\star)\,dt. \quad \ldots (107)$$

Since this inequality holds with an arbitrary δ, we find for $\delta \to 0$,

$$\underline{\lim}_{\tau\to\tau^\star} \Lambda(B)_\tau \geqq \Lambda(B)_{\tau^\star},$$

i e. the lower semicontinuity.

Case (*b*) Omitting the star in $B^\star$ we have, according to $f(\tau) \sim \tau$, to prove that a solution $B(\tau)$ of (83) satisfying

$$B \geqq 0, \quad \lim_{\nu=\infty} \frac{B(\tau_\nu)}{\tau_\nu} = 0, \quad \tau_\nu \to \infty, \quad \ldots (108)$$

vanishes identically According to $B \geqq 0$ and to $E_3 < E_1$, we have

$$\int_0^\infty B E_3(|\tau - t|)\,dt \leqq 2\Lambda(B)_\tau = 2B(\tau) \qquad ..(109)$$

$B(\tau)$, being a non-negative solution of (83), determines a solution of Problem Ia The corresponding net flux F is not known, however, we know from § 6, that necessarily $F \geqq 0$. From (56), $J = B$, and from (108), (109) we infer that

$$F \leqq 4 \lim_{\nu=\infty} \frac{B(\tau_\nu)}{\tau_\nu} = 0,$$

thus yielding $F = 0$ On inserting this value into (54), $J = B$, incident radiation = zero, we find for $\tau = 0$

$$\int_0^\infty B(t)\,E_2(t)\,dt = 0,$$

this being compatible with $B \geqq 0$ only if $B(\tau) \equiv 0$, which completes the proof of the uniqueness

COROLLARY. Every solution of (83) with a finite lower bound has the form const $f(\tau)$

Proof In the solution $B(\tau) + cf(\tau)$ the constant c can be chosen so large that this solution is everywhere positive Theorem V then completes the proof.

§ 11. A DIFFERENTIATION FORMULA

For the continuation of the discussion of Problems I and II, a study of the more general operator

$$\Lambda(f)_\tau \equiv \int_0^\infty f(t)\,H(|\tau - t|)\,dt \qquad ......(110)$$

and of the related integral equation will be useful. We introduce the notation

$$(g, h) = \int_0^\infty g(t)\,h(t)\,dt.$$

The kernel of (110) is evidently symmetrical The symmetry can also be expressed by the general relation

$$(g, \Lambda(h)) = (h, \Lambda(g)) = \int_0^\infty \int_0^\infty H(|\tau - t|)\,g(t)\,h(\tau)\,dt\,d\tau. \qquad ......(111)$$

We start with an important property of the particular operator (110). For this purpose let us suppose, that $H(x)$ is continuous for $x > 0$ and satisfies the conditions

$$H(x) = O\left(\log\frac{1}{x}\right),\ x \to 0, \quad H(x) = O(e^{-x}),\ x \to \infty \qquad \ldots\ldots(112)$$

Let $h(x)$ be continuous for $x \geqq 0$ and continuously differentiable for $x > 0$. $h(x)$ is, furthermore, supposed to have the properties

$$h'(x) = O\left(\log\frac{1}{x}\right),\ x \to 0, \quad h'(x) = O(e^{\theta x}),\ x \to \infty, \qquad \ldots\ldots(113)$$

with a suitable $\theta < 1$. Under these conditions, the formula

$$\frac{d}{d\tau}\Lambda(h)_\tau = \Lambda(h')_\tau + h(0)\,H(\tau) \qquad \ldots(114)$$

holds.

Proof We have

$$\Lambda(h)_\tau = \int_0^\tau H(t)\,h(\tau - t)\,dt + \int_0^\infty H(t)\,h(\tau + t)\,dt. \ \ldots\ (115)$$

On writing $S_1(\tau)$ for the first integral, we find

$$\frac{S_1(\tau+\delta) - S_1(\tau)}{\delta} = \frac{1}{\delta}\int_\tau^{\tau+\delta} H(t)\,h(\tau+\delta-t)\,dt$$
$$+ \int_0^\tau H(t)\,\frac{h(\tau+\delta-t) - h(\tau-t)}{\delta}\,dt. \ \ldots\ldots(115')$$

Here, the first term obviously tends to $h(0)\,H(\tau)$ as $\delta \to 0$. In the second term, however, limit process and integration may be interchanged. The proof of this follows the same line as the proof that a Newtonian potential can be differentiated under the integral sign, even when the moving point lies within the mass That the second integral in (115) may be differentiated under the integral sign, follows from a well-known theorem, according to which this is always permitted within a certain τ-interval if the τ-derivative of the integrand lies, in absolute value, below a fixed integrable function of t This test applies here since, according to (112) and (113),

$$|H(t)\,h'(\tau+t)| < \text{const.}\, e^{\theta\tau} e^{-(1-\theta)t}$$

holds for all sufficiently large t, the constant being independent of τ. Within an arbitrary finite τ-interval, having $\tau=0$ as an outside point, the suppositions of the above theorem are fulfilled because the absolute value in question is less than const $e^{-(1-0)t}$ for large t and less than const. $|\log t|$ for small t

§ 12 ASYMPTOTICALLY LINEAR SOLUTIONS OF $f=\Lambda(f)$

Since the formula of § 11 will not be used until § 17 we abandon the hypothesis (112) about the kernel. Let us consider the integrals

$$H_1(x)=H(x), \quad H_{n+1}(x)=\int_x^\infty H_n(v)\,dv.$$

We now suppose that the kernel is positive,

$$H(v)>0, \qquad \text{.(116)}$$

and that

$$H(x)\geqq\alpha H_2(x) \qquad \text{....(117)}$$

holds with suitable $\alpha>0$ All these assumptions are fulfilled in the case (52), with $H=\frac{1}{2}E_1$. (117) implies the existence of H_2, and automatically of all H_n

The integral equation $f(\tau)=\Lambda(f)_\tau$ ......(118)

will be studied hereafter by means of Fourier integrals It is, however, worthwhile to consider it first from the same, physically more natural, point of view as in § 10. (118) can have solutions of various asymptotic behaviour for τ large. Under the particular hypothesis

$$H_2(0)=\tfrac{1}{2}, \qquad \text{......(119)}$$

for instance, we have the

THEOREM VI Under the suppositions made above about the kernel, in particular (119), (118) possesses a solution

$$f(\tau)=\tau+q(\tau), \quad 0<q<\frac{1}{\alpha},$$

where α is taken from (117)

Proof. In Milne's case, $H=\frac{1}{2}E_1$, (119) is, according to $H_2(0)=E_2(0)/2=\frac{1}{2}$, certainly fulfilled. We first note that

$$\int_x^\infty tH_n(t)\,dt=xH_{n+1}(x)+H_{n+2}(x). \qquad \text{......(120)}$$

Taking account of (119), we find, after simple calculations, the relations

$$1 = \Lambda(1)_\tau + H_2(\tau) \qquad \ldots .(121)$$

and

$$\tau = \Lambda(t)_\tau - H_3(\tau). \qquad \ldots .(122)$$

According to (117), we obtain from (121) and (122) the inequalities

$$f_1(\tau) < \Lambda(f_1)_\tau, \quad f_1 = \tau,$$

and

$$\bar{f}(\tau) \geqq \Lambda(\bar{f})_\tau, \quad \bar{f} = \tau + \frac{1}{\sigma}.$$

This shows, in the same way as in § 10, that (118) has a solution between τ and $\tau + 1/\sigma$, q e d

From (118) and (122) we find the integral equation

$$q(\tau) = \Lambda(q)_\tau + H_3(\tau) \qquad \ldots ..(123)$$

for the remainder function $q(\tau)$ We must now remember that the above solution $f(\tau)$ of (118) was found to be the limit function of a sequence $f_n(\tau)$ defined by the recurrent relations

$$f_{n+1}(\tau) = \Lambda(f_n)_\tau, \quad f_1 = \tau$$

On setting $f_n = \tau + q_n$, we find

$$q_{n+1}(\tau) = \Lambda(q_n)_\tau + H_3(\tau), \quad q_1 = 0,$$

thus yielding

$$q(\tau) = \sum_0^\infty \Lambda^\nu (H_3)_\tau,$$

Λ^n being the nth iterate of the operator Λ, $\Lambda^0(f) = f$. In other words, $q(\tau)$ is represented by the Neumann series of the integral equation (123).

Definition If the solution of a linear integral equation is represented by the Neumann series, we call it the N-solution of that equation.

The function $q(\tau)$ of Theorem VI is accordingly the N-solution of (123)

Lemma 1. A nowhere negative solution of the inhomogeneous equation

$$\phi(\tau) = \Lambda(\phi)_\tau + \Phi(\tau), \quad \Phi \geqq 0 \qquad \ldots\ldots(124)$$

is always the sum of the N-solution of (124) and a nowhere negative solution of the homogeneous equation

Proof From (124) we have, according to the positivity of Λ,

$$\phi(\tau)=\sum_0^n \Lambda^\nu(\Phi)_\tau+\Lambda^{n+1}(\phi)_\tau \geqq \sum_0^n \Lambda^\nu(\Phi)_\tau .$$

Proceeding to the limit $n=\infty$, we see that the existence of a non-negative solution of (124) implies the convergence of the N-series of (124), this series consisting of nowhere negative terms The lemma follows then from the inequality

$$\phi(\tau) \geqq \sum_0^\infty \Lambda^\nu(\Phi)_\tau$$

LEMMA 2 A continuous solution g of (118) with the property $g/f \to c$, $\tau \to \infty$, necessarily equals cf

Proof. From the supposition made in the lemma we readily infer that the function

$$g(\tau)/f(\tau)$$

attains either its maximum or its minimum value for a finite value of τ, say $\tau^\star$ Without loss of generality we can confine ourselves to the case of the maximum value M. In this case we have

$$Mf(\tau)-g(\tau) \geqq 0, \quad Mf(\tau^\star)-g(\tau^\star)=0.$$

The left-hand side is here a solution of (118). According to the positivity of Λ, however, a non-negative solution of (118) can only vanish somewhere if it vanishes identically. It follows at the same time that $M=c$, q e d.

Let us now apply these simple results to the equation (121), having 1 as a solution All requirements of Lemma 1 are fulfilled in this case 1 is therefore nowhere smaller than the N-solution of (121) The N-solution being continuous everywhere, the difference is an evidently bounded and continuous solution of the homogeneous equation (118). If this solution is identified with the $g(\tau)$ of Lemma 2, $f(\tau)$ being the solution of Theorem VI, we find from that lemma that $g(\tau)$ vanishes identically. 1 is therefore the N-solution of the equation (121) We collect these facts in the

LEMMA 3. $q(\tau)$ of Theorem VI is the N-solution of (123) 1 is the N-solution of (121)

§ 13. AN AUXILIARY THEOREM WITH APPLICATION TO PROBLEM II

Let Λ, for a moment, be an arbitrary symmetrical operator For the solutions of the simultaneous integral equations

$$\phi_1 = \Lambda(\phi_1) + \Phi_1, \quad \phi_2 = \Lambda(\phi_2) + \Phi_2, \qquad .(125)$$

we find purely formally

$$(\phi_1, \Phi_2) = (\phi_1, \phi_2) - (\phi_1, \Lambda(\phi_2)), \quad (\phi_2, \Phi_1) = (\phi_2, \phi_1) - (\phi_2, \Lambda(\phi_1)),$$

and according to the symmetry of Λ,

$$(\phi_1, \Phi_2) = (\phi_2, \Phi_1) \qquad .....(126)$$

This formula is, of course, always correct when the integration interval is finite and when the kernel is continuous. The proof fails, however, in all of our cases, since here (ϕ_1, ϕ_2) is always infinite. Nevertheless, formula (126) holds also for certain singular integral equations, of course under certain restrictions.

AUXILIARY THEOREM Let Λ be a positive and symmetrical integral operator Furthermore, let us suppose that $\Phi_1 \geqq 0$, $\Phi_2 \geqq 0$ in (125) Formula (126) then holds, if in (125) ϕ_1 and ϕ_2 be the N-solutions

Proof. The iterated operators Λ^ν are well known to be symmetrical once Λ has this property From

$$\phi_1 = \sum_0^\infty \Lambda^\nu(\Phi_1), \quad \phi_2 = \sum_0^\infty \Lambda^\nu(\Phi_2),$$

we have

$$(\Phi_2, \phi_1) = \sum_0^\infty (\Phi_2, \Lambda^\nu(\Phi_1)) = \sum_0^\infty (\Phi_1, \Lambda^\nu(\Phi_2)) = (\Phi_1, \phi_2).$$

The termwise integration of an infinite series of functions, applied here, is readily seen to be justified, all terms being nowhere negative.

We now return to the special operator (110) and prove the

THEOREM VII. The necessary and sufficient condition that, under the suppositions (116), (117), (119),

$$\phi(\tau) = \Lambda(\phi)_\tau + \Phi(\tau), \quad \Phi \geqq 0 \qquad ......(127)$$

have a nowhere negative solution ϕ, is that

$$(1, \Phi) = \int_0^\infty \Phi(t)\,dt$$

be finite

Corollary. Under the same assumptions, Φ however having any sign, (127) has always a solution if

$$\int_0^\infty |(\Phi(t)|\,dt$$

be finite The N-series converges absolutely

Proof According to (117) and to the fact that $H_2(v)$ is a decreasing function, we have

$$H(t-\tau) \geqq \alpha H_2(t-\tau) > \alpha H_2(t),$$

thus yielding, with respect to $\phi \geqq 0$,

$$\int_\tau^\infty \phi(t)\,H(t-\tau)\,dt \geqq \alpha \int_\tau^\infty \phi(t)\,H_2(t)\,dt \quad \ldots\ldots(128)$$

Furthermore, we have $H(\tau - t) \geqq \alpha H_2(\tau - t)$ and, since $H_2(x)$ lies between positive limits within $(0, \tau)$,

$$\int_0^\tau \phi(t)\,H(\tau - t)\,dt \geqq \text{const} \int_0^\tau \phi(t)\,H_2(t)\,dt, \quad \ldots\ldots(129)$$

the constant being positive and depending continuously upon τ $\Lambda(\phi)_\tau$ must, now, be finite for some values of τ Hence, and from (128), (129), we may infer that (ϕ, H_2) is finite According to Lemma 1 this holds a fortiori, if ϕ is replaced by the N-solution of (127). On applying the auxiliary theorem to the two equations (121) and (127) and on taking account of Lemma 3, we find that

$$(1, \Phi) = (\phi, H_2) < \infty.$$

The condition is sufficient. On setting

$$\phi_n(\tau) = \sum_0^n \Lambda^\nu(\Phi)_\tau,$$

we infer from the symmetry of Λ^ν and from (121) that

$$\begin{aligned}(\phi_n, H_2) &= \Sigma(\Lambda^\nu(\Phi), H_2) = \Sigma(\Phi, \Lambda^\nu(H_2)) \\ &= (\phi, \Sigma\Lambda^\nu(H_2)) \leqq (\Phi, 1),\end{aligned}$$

proving thus that (ϕ_n, H_2) lies below a finite limit independent of n The fact that ϕ_n form an increasing set of functions

$$\phi_n \leqq \phi_{n+1}, \quad n = 1, 2, 3,$$

implies the existence of the limit function $\phi(\tau) = \lim \phi_n(\tau)$, and we infer that

$$(\phi, H_2) = \lim (\phi_n, H_2)$$

is a finite quantity At the same time we readily infer that $(\phi, H_2) = (\Phi, 1)$

The corollary simply follows from the theorem, since an absolutely integrable function $\Phi(\tau)$ can be written in the form $\Phi_1 - \Phi_2$, Φ_1 and Φ_2 being both non-negative and integrable.

Application to Problem II Milne's integral equation (53) is in the case of Problem II, $J = B$,

$$B(\tau) = \Lambda(B)_\tau + \frac{\epsilon}{\alpha} \qquad \text{......(130)}$$

Supposing that everywhere $\epsilon \geqq 0$, we find from Theorem VII that the necessary and sufficient condition that (130) have a non-negative solution is, that

$$4\pi \int_0^\infty \frac{\epsilon}{\alpha} dt, \qquad \text{......(131)}$$

i.e. the energy liberated per unit time in a normal column of cross-section one, be finite We prove, furthermore, the

THEOREM VIII Suppose that $\epsilon \geqq 0$ everywhere The necessary and sufficient condition that Problem II*a* has a solution is that (131) be finite The uniquely determined solution is given by the N-solution of (130)

Proof. Only the assertion concerning the N-solution of (130) needs to be proved We remember that Problem II*a* is the special case of Problem II where the surface flux $\pi \overline{F}_0$ satisfies

$$\overline{F}_0 = 4 \int_0^\infty \frac{\epsilon}{\alpha} dt \qquad \text{......(132)}$$

It is therefore to be proved that the N-solution $\bar{B}(\tau)$ of (130) automatically satisfies (132). From (54), $\tau = 0$, $J = \bar{B}$, taken without incident radiation, we find

$$\overline{F}_0 = 2(\bar{B}, E_2).$$

$\bar{B}$ being the N-solution of (130), we get from the auxiliary theorem, applied to (121) and (130),

$$(\phi, H_2) = \tfrac{1}{2}(\bar{B}, E_2) = (1, \Phi) = \left(1, \frac{\epsilon}{\alpha}\right),$$

which completes the proof of (132). For any other non-negative solution B of (130), however, the left-hand side of (132) would be greater than the right-hand side,

$$F_0 > 4\int_0^\infty \frac{\epsilon}{\sigma}\,dt \qquad \ldots.(133)$$

In order to obtain the solution of the general Problem II, we must remember formula (45′) giving F_∞ when F_0 is given

THEOREM IX Suppose that $\epsilon \geqq 0$ everywhere The solution of Problem II is uniquely determined by

$$B(\tau) = \bar{B}(\tau) + \tfrac{3}{4}F_\infty f(\tau),$$

F_∞ being determined from (45′), $\bar{B}$ being the N-solution of (130), and $f(\tau)$ being the function of Theorem IV

The uniqueness of the solution follows easily from the fact that any non-negative solution of (130) is the sum of the N-solution of (130) and a non-negative solution of the homogeneous equation, the latter being characterized by Theorem V. If $\pi\bar{F}$ denotes the flux in Problem IIa, we have

$$F_\tau = \bar{F}_\tau + F_\infty, \quad \bar{F}_\tau = 4\int_\tau^\infty \frac{\epsilon}{\sigma}\,dt, \quad F_\infty \geqq 0 \ldots\ldots(134)$$

If the assumption $\epsilon \geqq 0$ (everywhere) is abandoned, similar theorems hold. But we have then to formulate the general conditions under which the solution becomes non-negative as required in the physical problem

§ 14. OTHER APPLICATIONS
THE BOUNDARY TEMPERATURE

We set
$$g_\delta(\tau) = \frac{f(\tau+\delta) - f(\tau)}{\delta}, \qquad \ldots..(135)$$

where $f(\tau)$ is the solution of (118) mentioned in Theorem VI, and

$$G_\delta(\tau) = \frac{1}{\delta}\int_\tau^{\tau+\delta} H(t) f(\tau+\delta-t)\,dt, \qquad \ldots\ldots(136)$$

δ being a positive number From the integral equation (118) and from the identity (115), we find

$$g_\delta(\tau) = \Lambda(g_\delta)_\tau + G_\delta(\tau) \qquad \ldots (137)$$

Proceeding to the limit $\delta \to 0$ we get, purely formally, the integral equation for the derivative $f'(\tau)$ However, as we do not know if this derivative exists, we had better use (137) as it stands

In order to prove that g_δ is the N-solution of (137), we first note that

$$g_\delta(\tau) = \sum_0^n \Lambda^\nu (G_\delta)_\tau + \Lambda^{n+1}(g_\delta)_\tau \qquad .(138)$$

Keeping δ fixed we infer from Theorem VI that $g_\delta(\tau)$ is a bounded function of τ, $|g_\delta| < C$, whence

$$|\Lambda^{n+1}(g_\delta)_\tau| < C\Lambda^{n+1}(1)_\tau.$$

According to Lemma 3, we have

$$\Lambda^{n+1}(1) = \tfrac{1}{2}\sum_{n+1}^\infty \Lambda^\nu(E_2),$$

showing thus that the remainder in (138) tends to zero as $n \to \infty$. $g_\delta(\tau)$ is therefore the N-solution of (137).

THEOREM X The value of the solution $f(\tau)$ of (118), given in Theorem VI, is determined by a simple formula,

$$f_0 = \sqrt{2H_4(0)}$$

Proof Applying the auxiliary theorem of §13 to (121) and (123), we obtain

$$(q, H_2) = (1, H_3) = H_4(0). \qquad \ldots\ldots (139)$$

Since, by partial integration, $(t, H_2) = H_4(0)$, we find altogether, according to $f = \tau + q$,

$$(f, H_2) = 2H_4(0). \qquad \ldots\ldots (139')$$

Since, furthermore, g_δ is the N-solution of (137), the auxiliary theorem can be applied to (123) and (137), yielding thus

$$(q, G_\delta) = (g_\delta, H_3) \qquad \ldots\ldots (140)$$

On the other hand, we find by means of (135)

$$(g_\delta, H_3) = -\frac{1}{\delta}\int_0^\delta f(t)\,H_3(t)\,dt + \int_\delta^\infty f(t)\,\frac{H_3(t-\delta) - H_3(t)}{\delta}\,dt$$

The fraction in the second integral lies between the limits

$$H_2(t-\delta), \quad H_2(t)$$

because $H_2 = -H_3'$ is a decreasing function We may therefore proceed to the limit under the integral sign, finding thus

$$\lim_{\delta=0} (g_\delta, H_3) = -f_0 H_3(0) + (f, H_2) \qquad \text{.. (141)}$$

Furthermore, we have from (136)

$$\lim_{\delta=0} G_\delta(\tau) = f_0 H(\tau)$$

On inserting the right-hand integral in (136) into the left-hand side of (140) it is seen without difficulty that we may proceed to the limit $\delta \to 0$ under the integral sign on the left-hand side of (140),

$$\lim_{\delta=0} (q, G_\delta) = f_0 (q, H). \qquad \text{....(142)}$$

Finally, we have from (123), $\tau = 0$, according to $q_0 = f_0$,

$$f_0 = (q, H) + H_3(0) \qquad \text{.(143)}$$

Combination of (140), (141), (142) and (143) leads to

$$f_0^2 = (f, H_2),$$

whence, according to (139'), the required formula follows

LEMMA 4 Let $H(\tau)/H_2(\tau)$ be bounded for x large enough. Under the condition (112), the solution $f(\tau)$ of (118) has then a positive derivative $f'(\tau)$, being the N-solution of the integral equation

$$f'(\tau) = \Lambda(f')_\tau + f_0 H(\tau) \qquad \text{.. .(144)}$$

Proof. First we note that this applies to Milne's case, $H = \frac{1}{2} E_1$, since $E_1/E_2 \to 1$ as $x \to \infty$. Since the lemma is only needed for the proof of Theorem XI we shall content ourselves with brief indications of the rather lengthy proof. The main part of the proof consists in establishing an inequality for the N-solution of the integral equation

$$g(\tau) = \Lambda(g)_\tau + f_0 H(\tau). \qquad \text{. .(145)}$$

The proof is complicated by the possibility of an infinite $H(0)$ (this is just the case with Milne's equation) It is therefore convenient to split $H(\tau)$ into two parts

$$f_0 H = \Phi_1 + \Phi_2, \qquad \text{......(146)}$$

where

$$\phi_1 = \begin{cases} f_0 H, & 0 < \tau < a, \\ 0, & \tau \geqq a. \end{cases} \qquad \text{......(147)}$$

$\Phi_2(\tau)$ is zero for small τ and coincides with $f_0 H(\tau)$ for large τ. We consider then the two equations

$$\phi_1 = \Lambda(\phi_1) + \Phi_1, \quad \phi_2 = \Lambda(\phi_2) + \Phi_2 \qquad (148)$$

separately According to the supposition made in the lemma we have $\Phi_2(\tau) < \text{const}\ H_2(\tau)$ for all τ. The N-solution of the second equation exists therefore as seen by comparing it with the equation (121) At the same time we find

$$0 < \phi_2(\tau) < \text{const.}$$

for all τ As to the first equation, we begin with considering the expression

$$\Lambda(\Phi_1)_\tau = \int_0^a H(\tau - t)\, H(t)\, dt, \quad \tau > a$$

We note that $\Lambda(\Phi_1)$ is a continuous function of τ This holds, according to (112), also at $\tau = 0$ For τ large we find, according to the hypothesis made, and since H_2 decreases,

$$\Lambda(\Phi_1)_\tau < \text{const.} \int_0^a H_2(\tau - t)\, H(t)\, dt < \text{const}\ H_2(\tau - a), \quad (149)$$

the constant being independent of a Now we have, for τ large,

$$H_2(\tau - a) - H_2(\tau) = \int_{\tau - a}^{\tau} H\, dt$$

$$< \text{const.} \int_{\tau - a}^{\tau} H_2\, dt < a \,.\, \text{const.}\, H_2(\tau - a)$$

Since here the constant is again independent of a, we may choose once for all a so small that a const $< \frac{1}{2}$. For this value of a, we find then

$$H_2(\tau - a) < 2H_2(\tau),$$

thus yielding, with respect to (149),

$$\Lambda(\Phi_1)_\tau < \text{const}\ H_2(\tau)$$

for τ large The left-hand side being continuous for all τ, we can choose here the constant such that the inequality holds for all $\tau \geqq 0$. Comparing again with (121), we find that the N-series

$$\sum_1^\infty \Lambda^\nu(\Phi_1)_\tau$$

converges for all τ. On adding here the term $\nu = 0$, namely Φ_1, we obtain the complete N-solution ϕ_1 of the first equation (148).

Altogether we see that the N-solution $g(\tau)$ of (145) satisfies an inequality

$$g(\tau) < H(\tau) + \text{const}$$

From (137) and (145) we now find

$$g_\delta - g = \Lambda(g_\delta - g) + (G_\delta - f_0 H)$$

The second term on the right is easily seen to have the following properties as $\delta \to 0$. It is $O\left(\log \frac{1}{\tau}\right)$ for all small τ, uniformly with respect to δ, while

$$\frac{G_\delta(\tau) - f_0 H(\tau)}{H_2(\tau)}$$

tends to zero uniformly for all $\tau \geqq \tau_0$, τ_0 being an arbitrary number greater than zero. A similar splitting leads then to the result that g is the limit of g_δ, i e that $g(\tau) = f'(\tau)$ At the same time we find an inequality

$$f'(\tau) < H(\tau) + \text{const} \qquad \ldots (150)$$

The above considerations show that f' is the N-solution of (144).

THEOREM XI Suppose that $\Phi(\tau)$ is continuous at $\tau = 0$ Under the hypothesis made in Lemma 4, the N-solution of the integral equation (127) has, at $\tau = 0$, the value

$$\phi_0 = \Phi_0 + \frac{1}{f_0}(f', \Phi). \qquad \ldots\ldots(151)$$

Proof Since, under the stricter hypothesis about H made in Lemma 4, the derivative $f'(\tau)$ exists, we can dispense with the complicated formulae (135), (136), (137) and apply the auxiliary theorem of § 13 directly to (127) and (144), getting thus

$$(\Phi, f') = f_0(\phi, H)$$

From (127), $\tau = 0$, we find $(\phi, H) = \phi_0 - \Phi_0$, whence (151) follows. The reader will notice that (151) can also be written in the form of a Stieltjes-integral

$$\phi_0 = -\frac{1}{f_0}\int_0^\infty f(t)\, d\Phi(t) \qquad \ldots\ldots(152)$$

It is, of course, easy to obtain Theorem X from (152). Application to (123) yields, according to $q_0 = f_0$,

$$f_0^2 = -\int_0^\infty f\, dH_3 = (f, H_2),$$

which together with (139') proves the theorem.

Application to the boundary temperature in Problems I and II
In the case of Problem Ia we have

$$2H_4(0)=E_4(0)=\tfrac{1}{3},$$

yielding thus
$$f_0=\frac{1}{\sqrt{3}} \qquad \ldots\ldots (153)$$

for the solution mentioned in Theorem IV According to (84) we therefore get

$$B_0=\frac{\sqrt{3}}{4}F. \qquad \ldots (154)$$

Under local thermodynamical equilibrium we thus find, according to (98), the value

$$T_0^4=\frac{\sqrt{3}}{4}T_e^4$$

for the surface temperature of model Ia, independent of the variation of the absorption coefficient within the slab

Let us, now, find a formula for the boundary temperature of the model IIa. In order to apply Theorem IX we first suppose that $\epsilon \geqq 0$ everywhere We find for the N-solution $\bar{B}(\tau)$ of (130)

$$\bar{B}_0-\frac{\epsilon}{\alpha}\bigg|_0=\frac{1}{f_0}\int_0^\infty \frac{\epsilon}{\alpha}f'dt=\frac{1}{f_0}\int_0^\infty \frac{\epsilon}{\alpha}\,dt+\frac{1}{f_0}\int_0^\infty \frac{\epsilon}{\alpha}q'dt.$$

According to (132) and (153), the first term on the right equals

$$\frac{\sqrt{3}}{4}\bar{F}_0$$

From Theorem IX we therefore obtain, on account of $F_0=\bar{F}_0+F_\infty$,

$$B_0=\frac{\epsilon}{\alpha}\bigg|_0+\frac{\sqrt{3}}{4}F_0+\sqrt{3}\int_0^\infty \frac{\epsilon}{\alpha}q'dt, \qquad \ldots\ldots (155)$$

for the determination of the surface temperature of model II.

It should be noticed that (155) holds also in the general case, $\epsilon \gtrless 0$, provided that

$$\int_0^\infty \frac{|\epsilon|}{\alpha}\,dt=\int |\epsilon|\,\rho\,dx,$$

integrated through the slab, is finite In the next section we shall prove that $q(\tau)$ increases with increasing τ. Anticipating this result, we infer from (155) that, in the case $\epsilon \geqq 0$ everywhere, the boundary temperature is greater than in strict radiative equilibrium, $\epsilon=0$, the surface flux πF_0 being prescribed.

This minimal property of the boundary temperature in the case of strict radiative equilibrium implies, conversely, the increasing of the remainder function $q(\tau)$. Otherwise the energy liberated could evidently be distributed in such a way that that property becomes violated

§ 15 PROOF THAT $q(\tau)$ INCREASES

Let us, for a moment, reconsider the hypotheses made in the general case treated in §§ 11–14 The hypothesis (112) concerning the behaviour of $H(x)$ for small x was only required for the proof of formula (114), of Lemma 4 and of Theorem XI (existence of $f'(\tau)$)

For the proof of the Theorems VI–X we needed, however, only the suppositions stated in (116), (117) and (119), the latter being of particular importance because it is responsible for the asymptotically linear character of the solution.

Let us, now, throughout this section, suppose that $H(x)$ is of the form

$$H(x)=\int_1^\infty e^{-sx}d\rho(s), \qquad \ldots\ldots(156)$$

$\rho(s)$ increasing with increasing s. (119) is satisfied if and only if

$$\int_1^\infty \frac{d\rho(s)}{s}=\frac{1}{2}. \qquad \ldots\ldots(156')$$

From
$$H_n(x)=\int_1^\infty e^{-sx}s^{1-n}\,d\rho(s),$$

we see that (117) is certainly fulfilled, with $\sigma=1$ The last integral is readily seen to converge also for $n\leqq 0$, making thus the formula $H_n'=-H_{n-1}$ valid for all $n \gtreqless 0$.

On applying Schwartz's inequality to H_n, in the same way as we have done in deriving (76), we find

$$H_n^2<H_{n-1}H_{n+1}$$

for all n, implying thus

$$\frac{d}{dx}\,\frac{H_{n+1}(x)}{H_n(x)}>0. \qquad \ldots\ldots(157)$$

The analogue of (78) subsists therefore in the present general case, the proof being the same,

$$\frac{d}{dx}\frac{H_n(x-a)}{H_n(x)} < 0, \quad x > a \qquad \dots .(158)$$

We now begin with the proof that, under the hypothesis (156), (156'), the remainder function $q(\tau)$ of Theorem VI increases with increasing τ

Transforming the integral operator Λ by partial integration, we find the identity

$$u(\tau) - \Lambda(u)_\tau \equiv u_0 H_2(\tau) + \int_0^\tau H_2(\tau - t)\, du(t) - \int_\tau^\infty H_2(t-\tau)\, du(t), \qquad .(159)$$

Stieltjes-integrals being employed because we do not wish to make any use of the differentiability of $f(\tau)$. (123) can therefore be written

$$\int_0^\tau H_2(\tau - t)\, dq(t) - \int_\tau^\infty H_2(t-\tau)\, dq(t) = H_3(\tau) - q_0 H_2(\tau) \quad \dots(160)$$

Let, in the sequel, $u(\tau)$ denote any positive and nowhere decreasing function that satisfies the integral inequality

$$u(\tau) \geqq \Lambda(u)_\tau + H_3(\tau) \qquad \dots..(161)$$

for all τ. Continuity of u is not required An example of a function $u(\tau)$ is furnished by $u = 1$, for in (121) we have $H_2 > H_3$ The set of values taken by all functions $u(\tau)$ at a particular point τ has a greatest lower bound $U = U(\tau)$ $u \geqq 0$ and (161) implies $U(\tau) \geqq H_3(\tau) > 0$ Furthermore, $U(\tau)$ is evidently nowhere decreasing From (161) and from $u \geqq U$ we see that

$$u(\tau) \geqq \Lambda(U)_\tau + H_3(\tau)$$

holds for all functions u, and therefore for their greatest lower bound U $U(\tau)$ is thus itself a function u, in fact, the smallest function of that class.

LEMMA U is continuous for $\tau \geqq 0$ In the inequality

$$U(\tau) \geqq \Lambda(u)_\tau + H_3(\tau) \qquad \dots...(162)$$

the equality sign takes place at all points not being inner points of a constancy interval of U $\tau = 0$ is included herein.

We postpone the lengthy proof of this main lemma to the end of the section and show first that U cannot have a constancy interval, thereby proving that U increases and satisfies the same integral equation as $q(\tau)$ This would, however, imply $U=q$, since a bounded solution of the homogeneous equation vanishes, according to Lemma 2 Supposing that U have a constancy interval $\alpha \leqq \tau \leqq \beta$, we find, according to the lemma, from (159) and (162)

$$U_0 - \frac{H_3(\tau)}{H_2(\tau)} + \int_0^{\alpha} \frac{H_2(\tau-t)}{H_2(\tau)}\,dU - \int_{\beta}^{\infty} \frac{H_2(t-\tau)}{H_2(\tau)}\,dU \begin{cases} =0, & \tau=\alpha, \\ \geqq 0, & \tau>\alpha, \end{cases} \quad . \quad (163)$$

τ lying between α and β This can, however, be shown to lead to contradiction According to (157), the second term decreases with increasing τ According to (158), and to $dU \geqq 0$, the third term certainly never increases. Furthermore, $H_2(t-\tau)/H_2(\tau)$ increases with increasing τ, the denominator decreasing and the numerator increasing The fourth term in (163) is therefore also a nowhere increasing function of τ Altogether we find that the whole left-hand side of (163) decreases, $\alpha<\tau<\beta$, in obvious contradiction to the right-hand side.

Proof of the lemma For a nowhere decreasing function U the limits $U(\tau-0)$ and $U(\tau+0)$ exist always. Since U is the smallest of all functions u, we find

$$U(\tau-0)=U(\tau),$$

since otherwise $u(\tau)=U(\tau-0) \leqq U(\tau)$ would be a smaller function u.

In order to prove the continuity of U we must show that $\eta=0$, where

$$2\eta = U(\tau_0+0)-U(\tau_0) \geqq 0. \qquad \ldots\ldots(164)$$

The right-hand side of (162) being a continuous function, we derive from $U(\tau) \geqq U(\tau_0+0)$, $\tau>\tau_0$,

$$U(\tau) \geqq \Lambda(U)_\tau + H_3(\tau)+\eta, \quad \tau_0<\tau<\tau_0+\delta, \ \ldots\ldots(165)$$

with a suitable $\delta>0$. On introducing the auxiliary function

$$h(\tau)=\begin{cases} 1, & \tau \text{ in } (\tau_0,\tau_0+\delta), \\ 0, & \tau \text{ not in } (\tau_0,\tau_0+\delta), \end{cases} \qquad \ldots\ldots(166)$$

we first find the obvious inequalities

$$h(\tau) < \begin{cases} \Lambda(h)_\tau + 1, & \tau \text{ in } (\tau_0, \tau_0 + \delta), \\ \Lambda(h)_\tau, & \tau \text{ not in } (\tau_0, \tau_0 + \delta) \end{cases} \quad \ldots\ldots (167)$$

It is geometrically evident that the function

$$u(\tau) = U(\tau) - \eta h(\tau) \quad . (168)$$

never decreases and that $u \geqq 0$ Furthermore, the second inequality (167) shows that, outside the interval $(\tau_0, \tau_0 + \delta)$, u satisfies (161). On the other hand, we infer from (165) and from the first inequality (167) that (161) is also satisfied by u within that interval u is therefore a member of the class introduced above, and $u \geqq U$, which obviously implies $\eta = 0$

It is immediately seen that, in (162), the equality sign must hold at $\tau = 0$ Otherwise we could make the value of $U(0)$ slightly smaller without affecting (162), in contradiction to the definition of the smallest u-function

A point τ shall be called a 'proper point' of $U(\tau)$ when $U(\tau') > U(\tau)$ holds for every $\tau' > \tau$ According to the continuity of U, an 'improper point' is thus either a left end point or an inner point of a constancy interval of U A point that does not lie inside of such an interval is therefore either proper or a left end point of such an interval In the latter case, and for $\tau > 0$, the point is certainly a right-hand limit point of proper points. According to the continuity of U it is thus sufficient to prove the equality sign in (162) for all proper points τ_0. We define a number $\eta \geqq 0$ by setting

$$2\eta = U(\tau_0) - \Lambda(U)_{\tau_0} - H_3(\tau_0) \quad \ldots\ldots (169)$$

and show that $\eta > 0$ is impossible As U is continuous, (165) must hold in a sufficiently small interval $(\tau_0, \tau_0 + \delta)$. This time we use another auxiliary function

$$h(\tau) = \begin{cases} U(\tau) - U(\tau_0), & \tau \text{ in } (\tau_0, \tau_0 + \delta), \\ 0, & \tau \text{ not in } (\tau_0, \tau_0 + \delta). \end{cases}$$

According to $0 < U \leqq 1$ we have $\eta < 1$. $h(\tau)$ is again seen to obey the inequalities (167). The function u, defined by (168), belongs then again to the above class, since u satisfies (161) and decreases nowhere (geometrical evidence) This completes the proof of the lemma.

§ 16 OTHER PROPERTIES OF $q(\tau)$

The equation (96) can be immediately generalized,

$$\int_{\tau}^{\infty} q(t)\, H_2(t-\tau)\, dt - \int_0^{\tau} q(t)\, H_2(\tau-t)\, dt = H_4(\tau) \quad ..(170)$$

This equation can be checked by direct differentiation, taking account of (123). There should be an additional integration constant on the right. On comparing, for $\tau=0$, with (139) we see, however, that it has the value zero The left-hand side of (170) can easily be transformed by partial integration, thus yielding

$$\int_0^{\infty} H_3(|\tau-t|)\, dq(t) = H_4(\tau) - q_0 H_3(\tau). \quad . \quad . \quad .(170')$$

From (157) and from $H_{n+1} < H_n$ follows the existence of the limit

$$\lambda = \lim_{x=\infty} \frac{H_n(x)}{H_{n+1}(x)} \geqq 1,$$

thus yielding $H_{n+1}(x) = e^{-(\lambda+\delta)x}$, where $\delta = \delta(x) \to 0$ as $x \to \infty$. According to (156) we have

$$H_{n+1}(x) > \int_1^{1+\epsilon} e^{-sx} s^{-n} d\rho(s) > e^{-(1+\epsilon)x} \int_1^{1+\epsilon} s^{-n} d\rho(s),$$

ρ increasing and ϵ being arbitrary, whence follows $\lambda \leqq 1$. This implies

$$\lambda = 1.$$

From (158) we infer, furthermore, the existence of

$$\lim_{x=\infty} \frac{H_n(x-y)}{H_n(x)} = \phi(y) \geqq 1.$$

From $H_n \sim H_{n-1} = -H_n'$ we obtain, by differentiation of this limit relation with respect to y, $\phi'(y) = \phi(y)$ This implies, together with $\phi(0)=1$, $\phi(y)=e^y$. According to (157) and (158), we find at the same time that

$$H_n(\tau-t) > e^t H_n(\tau), \quad \tau > t.$$

Since, now, $q(t)$ increases, we find from (170')

$$\int_0^{\tau} e^t dq(t) < \frac{H_4(\tau)}{H_3(\tau)} - q_0 < 1 - q_0,$$

thus yielding

$$\int_0^{\infty} e^t dq(t) \leqq 1 - q_0. \quad .....(171)$$

Let us, in (156), make an additional hypothesis concerning the behaviour of $\rho(s)$ at $s=1$. We suppose that

$$\int_1^\infty \frac{d\rho(s)}{s-1} = \infty. \qquad \text{.....} \ (172)$$

When ρ has a continuous derivative this is certainly true if $\rho'(0)>0$. This takes place, for instance, in Milne's case, $\rho(s)=\frac{1}{2}\log s$. Under this hypothesis we always have

$$\int_0^\infty e^t dq(t) = 1 - q_0. \qquad \text{.....}(173)$$

It is remarkable that this general relation does not contain H or ρ explicitly.

Proof of (173). We first show that the functions

$$l_n(x) = \int_0^x H_n(s)\, e^s ds \qquad \text{.....} \ (174)$$

have the properties

$$l_n(x) \to \infty, \ \frac{l_{n+1}(x)}{l_n(x)} \to 1; \quad x \to \infty. \qquad \text{.....} \ (174')$$

According to $H_{n+1}/H_n \to 1$ as $s\to\infty$, the second property is seen to be a consequence of the first one. Interchanging the order of integration we find

$$\int_0^\infty H_n(x)\, e^x dx = \int_1^\infty \frac{s^{1-n}}{1-s}\, d\rho(s),$$

which is infinite according to (172).

We need the inequality ($c=H_4(0)$)

$$\int_0^x e^\tau H_3(|\tau - t|)\, d\tau \leqq \begin{cases} e^t\{l_3(x)+c\}, & t\leqq x, \\ ce^t, & t>x. \end{cases} \qquad \text{.....}(174'')$$

For $t>x$ the left-hand side equals, indeed,

$$\int_0^x e^\tau H_3(t-\tau)\, d\tau < e^x \int_0^x H_3(t-\tau)\, d\tau < ce^x < ce^t.$$

For $t\leqq x$, however, it equals

$$\int_0^t e^\tau H_3(t-\tau)\, d\tau + \int_t^x e^\tau H_3(\tau - t)\, d\tau.$$

The first term is smaller than ce^{-t}, while the second term becomes, on substituting $\tau = t+s$,

$$e^t \int_0^{x-t} e^s H_3(s)\,ds,$$

being obviously less than $e^t l_3(x)$, which proves (174″) According to (174″) we get, on multiplying (170′) by e^τ and integrating through $0<\tau<x$,

$$l_4(x) - q_0 l_3(x) < \{l_3(x)+c\} \int_0^x e^t dq + \int_x^\infty e^t dq$$

Dividing through by $l_3(x)$ and proceeding to the limit $x \to \infty$, leads, according to (174′), to

$$1 - q_0 \leqq \int_0^\infty e^t dq,$$

whence, with respect to (171), (173) follows.

By partial integration, (173) can be transformed into

$$\int_0^\infty e^t \{q_\infty - q(t)\}\,dt = 1 - q_\infty. \qquad \ldots (173')$$

Numerical remarks on Milne's case. Let us first find the value of the constant a in the formula (97) The limit of the left-hand side as $\tau \to \infty$ evidently equals the limit of

$$\int_\omega^\infty q(t)\, E_3(|\tau - t|)\,dt,$$

ω being an arbitrarily fixed quantity. This integral is, now, included between the limits

$$q(\omega)\int_\omega^\infty E_3(|\tau-t|)\,dt, \quad q_\infty \int_\omega^\infty E_3(|\tau-t|)\,dt.$$

Here, the integral tends to $\frac{2}{3}$ as $\tau \to \infty$, whence

$$\tfrac{2}{3} q(\omega) \leqq a \leqq \tfrac{2}{3} q_\infty,$$

and according to the arbitrariness of ω,

$$a = \tfrac{2}{3} q_\infty \qquad \ldots (175)$$

From (97), $\tau = 0$, and from (175),

$$(q, E_3) = \tfrac{2}{3} q_\infty - \tfrac{1}{4}. \qquad \ldots (176)$$

Furthermore, from (96), $\tau = 0$, or from (139), $H = \frac{1}{2} E_1$,

$$(q, E_2) = \tfrac{1}{3} \qquad \ldots (177)$$

Practical computation of a number means including it between sufficiently narrow limits This can sometimes be done by means of mathematical artifices. The two relations (176), (177), together with the fact that q increases, suffice to find the value of q_∞ up to 1 per cent,

$$\tfrac{17}{24} < q_\infty < \tfrac{5}{7}, \tag{178}$$

or, in rounding off the second digit, $q_\infty = 0{\cdot}71$.

We have, in fact, from (176) and (177),

$$q_\infty = \tfrac{3}{2}(q, E_3 - \tfrac{1}{2}E_2) + \tfrac{5}{8},$$

which, according to $2E_3 > E_2$ for $\tau > 0$, implies

$$q_\infty < \tfrac{3}{2}q_\infty(1, E_3 - \tfrac{1}{2}E_2) + \tfrac{5}{8} = \tfrac{1}{8}q_\infty + \tfrac{5}{8},$$

i e the right-hand inequality of (178) On the other hand, from $3E_4 > 2E_3$,

$$\int_0^\infty (E_4 - \tfrac{2}{3}E_3)\, dq > 0$$

Since here the integrand vanishes at $t = 0$ and at $t = \infty$, we obtain by partial integration

$$(q, \tfrac{2}{3}E_2 - E_3) < 0.$$

Together with (176) and (177), this implies the second inequality of (178)

We now consider the law of darkening (101) and notice that the second term in the parenthesis decreases with increasing θ, $\theta < \pi/2$. Milne's first approximation consists in replacing q by a constant such that the emergent radiation gives the required net flux πF, thus yielding the value $\frac{2}{3}$ for the constant F being a definite average of the emergent radiation, we see that the second term lies, for θ near $\pi/2$, below Milne's approximation $\frac{2}{3}$ and rises, as θ decreases, above it The smallest value of the second term equals $q_0 = 1/\sqrt{3}$, while the greatest value (q, e^{-t}) surpasses $\frac{2}{3}$ We may add that

$$\tfrac{2}{3} < (q, e^{-t}) < 0{\cdot}69. \qquad \ldots\ldots(179)$$

We have, in fact, from (177) and from $E_2 < e^{-t}$,

$$(q, e^{-t}) = (q, e^{-t} - E_2) + \tfrac{1}{3} < \frac{q_\infty}{2} + \tfrac{1}{3} < 0{\cdot}69.$$

§ 17 PROBLEM I*b* FOR PARALLEL INCIDENT RADIATION

Parallel radiation carrying finite energy is to be considered as a limiting case. Let

$$r' = (\pi - \theta', \phi'), \quad \theta' < \frac{\pi}{2},$$

be the direction of an incident ray, and let $\Delta\omega$ be a solid angle of directions containing r'. We set

$$I(0, r) = \begin{cases} \pi S/\Delta\omega, & r \text{ in } \Delta\omega, \\ 0, & r \text{ not in } \Delta\omega, \end{cases}$$

for the incident radiation. When $\Delta\omega$ shrinks down to the single direction r', we speak of parallel radiation of the direction r' and of the flux

$$\pi S \geqq 0,$$

through the unit area normal to r'. On proceeding to the limit $\Delta\omega \to 0$, Milne's fundamental integral equation (53) becomes

$$B(\tau) = \Lambda(B)_\tau + \frac{S}{4} e^{-s\tau}, \quad s = \sec\theta'. \qquad \ldots(180)$$

We define (180) as the integral equation for parallel incident radiation of direction r'.

According to Theorem VII, (180) has a positive solution. Let us, in particular, consider the N-solution, i.e the smallest positive solution of (180). We put

$$B(\tau) = S g_s(\tau), \qquad ..(181)$$

g_s being the N-solution of (180) for $S = 1$

THEOREM XII. The N-solution of (180) can be explicitly expressed in terms of the solution $f(\tau) = \tau + q(\tau)$ of Problem I*a*,

$$g_s(\tau) = \tfrac{3}{4} s^2 (f, e^{-st}) \left\{ \frac{f(\tau)}{s} - \int_0^\tau e^{s(t-\tau)} f(t)\, dt \right\}. \qquad \ldots\ldots(182)$$

Proof. (182) can also be written in the form

$$g_s(\tau) = \tfrac{3}{4} s^2 (f, e^{-st}) \left\{ \frac{1 - e^{-s\tau}}{s^2} + \frac{q(\tau)}{s} - \int_0^\tau e^{s(t-\tau)} q(t)\, dt \right\} \ldots\ldots.(182')$$

First we show that the function

$$h(\tau)=\int_0^\tau e^{s(t-\tau)}f(t)\,dt \qquad \ldots \quad (182'')$$

solves (180) with a suitable S From (182''),

$$sh+h'=f, \quad h(0)=0. \qquad \ldots \quad (183)$$

Formula (114) is applicable, h being continuously differentiable and satisfying $h'=0(\tau)$ for τ large On setting

$$\rho(\tau)=h(\tau)-\Lambda(h)_\tau,$$

we find from (114) and from (183),

$$s\rho+\rho'=f-\Lambda(f)=0$$

Hence $\rho(\tau)=\text{const } e^{-s\tau}$. Now we show that in

$$g_s(\tau)=\Lambda(g_s)_\tau+ae^{-s\tau} \qquad \ldots \quad (184)$$

a equals $\frac{1}{4}$ It is obvious from (182') that $g_s(\tau)$ is a bounded function of τ The boundedness clearly implies that it is the N-solution of (184) Applying, now, formula (152) to (184), $\Phi=ae^{-s\tau}$, we find

$$g_s(0)=a\frac{s}{f_0}(f, e^{-st}).$$

If $g_s(0)$ is computed from (182), we obtain, according to $f_0=1/\sqrt{3}$, $a=\frac{1}{4}$. This completes the proof of our theorem

We note the special cases $\tau=0, \infty$, in (182),

$$g_s(0)=\frac{\sqrt{3}}{4}s(f, e^{-st}), \quad g_s(\infty)=\tfrac{3}{4}(f, e^{-st}), \qquad \ldots \quad (185)$$

thus yielding

$$\frac{g_s(0)}{g_s(\infty)}=\frac{s}{\sqrt{3}}, \qquad \ldots\ldots (186)$$

(185) being expressed in terms of the solution f of the Problem Ia. In particular we have the formula

$$\left(\frac{T_0}{T_\infty}\right)^4=\frac{\sec\theta'}{\sqrt{3}} \qquad \ldots \quad (187)$$

for the ratio of the limiting temperatures

The fundamental limit relation (56), $J=B$, $F_\infty=F$, is easily seen to remain valid in the case of parallel incident radiation. According to the boundedness of $B(\tau)$ the right-hand integral

represents also a bounded function of τ, whence $F=0$ We may collect the above results in the

THEOREM XIII The N-solution $B(\tau)=Sg_s(\tau)$ of (180) corresponds to vanishing net flux, $F=0$, i e it gives the solution of the Problem Ib (insolation problem) for parallel incident radiation of direction $\imath'=(\pi-\theta', \phi')$, $s=\sec\theta'$

By partial integration we find from (182′), according to (185), that $g_s(\tau)-g_s(\infty)$ equals

$$\lambda(\tau)=\int_0^\tau e^{st}dq(t)-(1-q_0)+1-\frac{1}{s},$$

up to a positive factor For $s\leqq 1$, we have, according to (173), $\lambda<0$ for all τ, whereas for $s>1$,

$$\lambda(0)=\frac{1}{\sqrt{3}}-\frac{1}{s}, \quad \lambda(\infty)>1-\frac{1}{s}>0.$$

$\lambda(\tau)$ being an increasing function, we infer that the equation $\lambda(\tau)=0$ has at most one root Furthermore, on differentiating (182′), and using (173), we find that, for $s\leqq 1$, $g_s(\tau)$ increases with increasing τ. Altogether we note the following behaviour of the solution $B(\tau)$ of (180):

$s\leqq 1$: $B(\tau)$ increases.

$\sqrt{3}>s>1$. $B(\tau)$ lies first below the value B_∞ and then rises above it

$s>\sqrt{3}$: $B(\tau)$ lies entirely above B_∞.

$s\leqq 1$ ($s=1$ corresponds to normal incidence) is the only case where B is monotonic

Generalization. Model Ib with parallel incident radiation, and with $F=0$, mainly applies to the upper layers of a planetary atmosphere. The hotter solar radiation will actually have a smaller absorption coefficient than the cooler radiation of the atmosphere The next idealizing step would thus consist in making the former coefficient a constant fraction n of the latter coefficient α.

In order to find the integral equation we must go back to the fundamental flux equation (54) The first term on the right is the only one that contains the radiation from outside In this term, therefore, τ must be replaced by $n\tau$ Proceeding as before to the limiting case of parallel radiation, we find, $J=B$,

$$F=S\cos\theta' e^{-n\tau\sec\theta'}+2\int_\tau^\infty BE_2(t-\tau)\,dt-2\int_0^\tau BE_2(\tau-t)\,dt,$$

$F=\text{const}$ On differentiating this relation and on dividing through by 4, we find

$$B(\tau)=\Lambda(B)_\tau+\frac{nS}{4}e^{-n\tau\sec\theta'}$$

Since this is of the same type as (180), all formulae of this section remain unchanged, with the only difference that S is to be replaced by nS and that s is now $n\sec\theta'$ We have, instead of (181) and (182),

$$B(\tau)=nSg_{n\sec\theta'}(\tau),\quad \frac{B_0}{B_\infty}=n\frac{\sec\theta'}{\sqrt{3}}.$$

§ 18. THE EMERGENT LIGHT

It will be convenient now to introduce the new notation

$$\Gamma_s(\sigma)=\sigma\int_0^\infty e^{-\sigma t}g_s(t)\,dt,\quad \Phi(\sigma)=\sigma\int_0^\infty e^{-\sigma t}f(t)\,dt \quad \dots(188)$$

The emergent radiation (law of darkening) becomes then

Problem Ia: $\quad I(0,\theta)=\frac{3}{4}F\Phi(\sec\theta), \quad \dots(189)$

and in the case of parallel incidence,

Problem Ib: $\quad I(0,\theta)=S\Gamma_{\sec\theta'}(\sec\theta) \quad \dots(189')$

From (183) we find, by partial integration,

$$\int_0^\infty e^{-\sigma t}h'\,dt=\sigma\int_0^\infty e^{-\sigma t}h\,dt,$$

thus yielding, again with respect to (183),

$$\Phi(\sigma)=\sigma(s+\sigma)\int_0^\infty e^{-\sigma t}h\,dt$$

This gives, together with (182) and (182″), the formula

$$\Gamma_s(\sigma)=\frac{3}{4}\frac{\sigma}{s+\sigma}\Phi(s)\Phi(\sigma). \qquad \ldots(190)$$

We have therefore the

THEOREM XIV In Problem I*b* for parallel radiation with the angle of incidence θ', the emergent radiation can be explicitly expressed in terms of the same radiation for Problem I*a*,

$$I(0,\theta)=\tfrac{3}{4}S\frac{\cos\theta'\Phi(\sec\theta)\Phi(\sec\theta')}{\cos\theta+\cos\theta'} \qquad \ldots\ (191)$$

In the general Problem I, with parallel incident radiation, but with an arbitrary flux $\pi F \geqq 0$, the emergent radiation is, of course,

$$I(0,\theta)=\frac{3}{4}\left[F+S\frac{\cos\theta'\Phi(\sec\theta')}{\cos\theta+\cos\theta'}\right]\Phi(\sec\theta).$$

This applies to binary stars, where the incident radiation is due to the other component.

§ 10 PROBLEM I*b* FOR ARBITRARY INCIDENT RADIATION

Since arbitrary incident radiation can be obtained by superposition of parallel bundles of different directions, we should be able to express the solution of Problem I*b*, for any given incident radiation, in terms of the solution for parallel incidence. For given incident radiation $I(0,r')>0$, a solution of (53), $\epsilon=0$, $J=B$, is, in fact, obviously given by

$$B(\tau)=\frac{1}{\pi}\int_{-}I(0,r')g_s(\tau)\,d\omega'$$

$$=\frac{1}{\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}I(0,\pi-\theta',\phi')\,g_{\sec\theta}(\tau)\sin\theta'\,d\theta'\,d\phi'. \qquad \ldots\ \ldots(192)$$

Let us show that this solution corresponds to vanishing net flux, $F=0$ We note that, for any value of τ, $g_s(\tau)$ decreases with increasing s This is physically evident, for it means that the temperature becomes smaller when the incident radiation is more

oblique Remember, in fact, that $g_s(\tau)$ is the N-solution of (180), $s=1$ The above assertion follows simply from the fact that the second term on the right of (180) decreases with increasing $s, s \geqq 1$. We have, in particular, $g_s(\tau) \leqq g_1(\tau)$, and from (192)

$$B(\tau) \leqq g_1(\tau) \quad \frac{1}{\pi} \int_{-} I(0, \imath') \, d\omega'.$$

The boundedness of $B(\tau)$ implies, as before, the vanishing of the net flux.

CHAPTER III

DISCUSSION OF PROBLEMS III AND IV

§ 20 SOLUTION OF PROBLEM III

Problem III has been analytically set up in § 7 We remember that the flux πF is constant and that the limit relation (44) holds, in consequence of the hypotheses, that the law of scattering is symmetric in the directions of the incident and scattered ray (11′), and that the same amount of radiation is scattered in opposite directions (11″).

THEOREM XV. Problem III always has a solution

$$J(\tau, r) = \tfrac{3}{4} F f(\tau, r), \qquad \ldots (193)$$

with the property

$$f(\tau, r) = \tau + q(\tau, r), \quad 0 < q < 1, \qquad \ldots (194)$$

the limits thus being independent of the law of scattering. Under the hypothesis

$$\gamma(\tau, r, r') > \text{const} > 0$$

the solution is unique

Proof The reasoning is essentially the same as in Problem I*a*, i e in the special case of uniform scattering, $\gamma \equiv 1$ We put

$$J_1(\tau, r) = \tau, \quad \bar{J}(\tau, r) = \tau + 1, \qquad \ldots (195)$$

and denote by $I_1(\tau, r)$ the respective intensities found from (47) and (57) in setting $J = J_1$, $J = \bar{J}$. On inserting these intensities into the right-hand side of (41) we obtain the expressions

$$\Lambda(J_1)_{\tau, r}, \quad \Lambda(\bar{J})_{\tau, r},$$

respectively.

Using the more convenient relations (99), (99′), instead of (47) and (57), we readily find

$$I_1(\tau, r) = \begin{cases} \tau + \cos\theta, & \theta < \pi/2, \\ \tau + \cos\theta + |\cos\theta|\, e^{-\tau|\sec\theta|}, & \theta > \pi/2, \end{cases}$$

and $$\bar{I}(\tau, r) = \begin{cases} \tau + 1 + \cos\theta, & \theta < \pi/2, \\ \tau + 1 + \cos\theta - (1 + \cos\theta)\, e^{-\tau|\sec\theta|}, & \theta > \pi/2 \end{cases}$$

These equations imply, in particular, the important inequalities

$$I_1(\tau, r) \geqq \tau + \cos\theta, \quad \bar{I}(\tau, r) \leqq \tau + 1 + \cos\theta,$$

for all directions $\imath = (\theta, \phi)$ Inserting this into (41) and taking into account the relation

$$\int \cos\theta' \gamma(\tau, \imath', r)\, d\omega' = 0,$$

we obtain the important inequalities

$$\Lambda(J_1)_{\tau,r} > \tau = J_1(\tau, \imath),$$
$$\Lambda(\bar{J})_{\tau,r} < \tau + 1 = \bar{J}(\tau, \imath).$$

According to the positivity of the operator Λ, we can, now, apply the same reasoning as in § 9 On setting

$$J_{n+1}(\tau, \imath) = \Lambda(J_n)_{\tau,r},$$

we infer that J_{n+1} lies above J_n, but that all J_n lie below $\bar{J}$. The limit function $f(\tau, r)$ is seen to satisfy the integral equation

$$f(\tau, r) = \Lambda(f)_{\tau,\imath}, \qquad \ldots\ldots(106)$$

and to lie between the indicated limits τ and $\tau+1$.

We set
$$J(\tau, r) = cf(\tau, r)$$

for the solution of J of Problem III and determine the constant c as in § 9 The intensity obviously lies between the limits

$$c(\tau + \cos\theta) < I(\tau, r) < c(\tau + 1 + \cos\theta), \quad \ldots\ldots(107)$$

for all directions r, whence

$$\lim_{\tau=\infty} \frac{I(\tau, \imath)}{\tau} = c$$

uniformly for all directions $\imath$. This gives, according to the fundamental relation (44),

$$c = \tfrac{3}{4} F. \qquad \ldots\ldots(108)$$

The proof of the uniqueness of the solution follows the same lines as in § 10 It is sufficient to repeat the main part of the proof, i e. to show that a solution J of (106) with the properties

$$J \geqq 0, \ \lim_{\nu=\infty} \frac{J(\tau_\nu, r_\nu)}{f(\tau_\nu, \imath_\nu)} = 0, \quad \tau_\nu \to \infty, \qquad \ldots\ldots(109)$$

τ_ν being a given sequence of numbers tending to infinity, vanishes identically

According to the hypothesis made about γ, and according to (41), we have

$$J = \frac{1}{4\pi} \int I\gamma\, d\omega' \geqq \text{const.} \int I \cos^2\theta'\, d\omega'.$$

On comparing this with (44) and (199) we get $F = F_\infty = 0$. The incident radiation (at $\tau = 0$) being zero we thus have, for $\tau = 0$,

$$F = F_+ = \frac{1}{\pi} \int_+ I(0, r) \cos\theta \, d\omega = 0,$$

which is possible only if the emergent radiation vanishes Thence, and from (99), $\tau = 0$, we infer that the Ergiebigkeit must vanish too, q e d.

Remarks on the uniqueness When the hypothesis made about the law of scattering is abandoned, Problem III could have several solutions This takes place, for instance, in the easily integrable limit case, where half the radiation is scattered backwards while the other half goes in the original direction. Under the hypothesis, however, that

$$\lim_{\tau=\infty} \frac{I(\tau, r')}{I(\tau, r)} = 1$$

holds uniformly for all directions, i.e that the radiation becomes isotropic at great depth, the solution is readily found to be unique For we find from (41), (44) and (194) that then

$$\lim_{\tau=\infty} \frac{J(\tau, r)}{f(\tau, r)} = \tfrac{3}{4} F$$

holds uniformly for all directions. As in the proof of Lemma 2, § 12, we infer, from the positivity of the operator, that $J \equiv \frac{3}{4} F f$.

§ 21. DISCUSSION OF PROBLEM IV IN THE CASE (65)

The optical thickness τ of the slab being finite, we consider the case (65) of constant radiation $I^\star$ incident at the inner face $\tau = \tau^\star$.

For the radiation coming from above we use again (99′), while the radiation from below is given by (61), or by the more convenient formula

$$I(\tau, r) = e^{-(\tau^\star - \tau)\sec\theta} I^\star + \int_0^{(\tau^\star-\tau)\sec\theta} e^{-s} J(r + s\cos\theta, r)\, ds, \quad \theta < \pi/2. \quad \ldots\ldots (200)$$

THEOREM XVI Problem IV, with the boundary condition (65), has a unique solution The Ergiebigkeit lies between the limits

$$I^\star \frac{\tau}{\tau^\star+1} < J(\tau, r) < I^\star \frac{\tau+1}{\tau^\star+1}, \quad .(201)$$

the limits thus being independent of the law of scattering

Proof This theorem has, in the case of uniform scattering, been proved by Schwarzschild

The integral equation (63) of Problem IV can be written in the form

$$J(\tau, r) = L(J)_{\tau,r} + H(\tau, r),$$

where

$$H(\tau, r) = \frac{I^\star}{4\pi} \int_{+} e^{-(\tau^\star - \tau)\sec\theta'} \gamma\, d\omega'. \quad . \quad (202)$$

We set $J_1(\tau, r) = I^\star \frac{\tau}{\tau^\star+1}, \quad \bar{J}(\tau, r) = I^\star \frac{\tau+1}{\tau^\star+1}$. .....(203)

In order to compute the quantities

$$L(J_1)_{\tau,r} + K(\tau, r), \quad L(\bar{J})_{\tau,r} + K(\tau, r),$$

we first find, from (99′) and (200), the intensities I_1 and $\bar{I}$ corresponding to the Ergiebigkeiten J_1 and $\bar{J}$, and then insert them into the right-hand side of (41) After some simple computations we get

$$I_1(\tau, r) = \frac{I^\star}{\tau^\star+1} \cdot \begin{cases} \tau + \cos\theta + (1-\cos\theta)\, e^{-(\tau^\star-\tau)\sec\theta}, & \theta < \pi/2, \\ \tau + \cos\theta + |\cos\theta|\, e^{-\tau|\sec\theta|}, & \theta > \pi/2, \end{cases}$$

and

$$\bar{I}(\tau, r) = \frac{I^\star}{\tau^\star+1} \begin{cases} \tau + 1 + \cos\theta - \cos\theta\, e^{-(\tau^\star-\tau)\sec\theta}, & \theta < \pi/2, \\ \tau + 1 + \cos\theta - (1+\cos\theta)\, e^{-\tau|\sec\theta|}, & \theta > \pi/2. \end{cases}$$

These relations imply, for all directions r, the important inequalities

$$I_1(\tau, r) \geqq I^\star \frac{\tau + \cos\theta}{\tau^\star+1},$$

$$\bar{I}(\tau, r) \leqq I^\star \frac{\tau + 1 + \cos\theta}{\tau^\star+1}.$$

On inserting the intensities into the right-hand integral of (41), we obtain, according to

$$\int \cos\theta' \gamma(\tau, r', r)\, d\omega' = 0,$$

the final inequalities

$$L(J_1)_{\tau,r}+H(\tau,r)>I^\star\frac{\tau}{\tau^\star+1}=J_1(\tau,r),\ \ldots\ ..(204)$$

$$L(\bar J)_{\tau,r}+H(\tau,r)<I^\star\frac{\tau+1}{\tau^\star+1}=\bar J(\tau,r)\quad \ldots ..(205)$$

According to the positivity of the operator L, we see as before that there is a solution between the indicated limits On setting

$$J_{n+1}(\tau,r)=L(J_n)_{\tau,r}+H(\tau,r),\qquad \ldots ..(206)$$

J_1 being defined by (203), we can write (204) in the form $J_2>J_1$. This implies $L(J_2)>L(J_1)$ and, according to (206), $n=1$, 2, $J_3>J_2$, and generally $J_{n+1}>J_n$ Furthermore, the relation

$$\bar J-J_{n+1}=L(\bar J-J_n)$$

and the inequality $\bar J-J_1>0$ show that generally $J_n<\bar J$ The limit function J of the functions J_n thus exists and has all the properties indicated in Theorem XVI. J is, moreover, seen to be the N-solution of (63)

In the present case of a finite $\tau^\star$, the uniqueness is inferred by a classical conclusion. First we note that

$$L(1)_{\tau,r}=1-\frac{1}{4\pi}\int_- e^{-\tau|\sec\theta'|}\gamma d\omega'-\frac{1}{4\pi}\int_+ e^{-(\tau^\star-\tau)\sec\theta'}\gamma d\omega';\qquad \ldots\ldots(207)$$

in particular, that $\qquad L(1)_{\tau,r}\leqq\lambda<1$

holds with a suitable constant λ. The difference $D(\tau,r)$ of two solutions of (63) satisfies the homogeneous equation $D=L(D)$. Denoting by M the maximum value of $|D|$, we find

$$|D|\leqq L(|D|)\leqq ML(1)\leqq\lambda M;$$

in particular $M\leqq\lambda M$, which, according to $\lambda<1$, gives $M=0$.

Large optical thickness. In the case of uniform scattering, Schwarzschild recognized, that for $\tau^\star\to\infty$, model IV goes over into model III. This holds, more generally, for any given law of scattering, of the types (11′) and (11″).

THEOREM XVII. For $I^\star = c\tau^\star$, c being fixed, the Ergiebigkeit $J(\tau, \tau^\star)$ tends towards $cf(\tau, \tau^\star)$ as the optical thickness $\tau^\star$ increases indefinitely

The main part of the proof is to show that the Ergiebigkeit $J(\tau, \tau^\star)$ tends towards a limit function as $\tau^\star \to \infty$ We may content ourselves with a brief sketch of the proof From the integral equation (63) it can be first recognized that the Ergiebigkeit is uniformly continuous for all $\tau^\star$ Arzela's selection theorem then guarantees the existence of one or several limit functions Such a limit function necessarily lies between $c\tau$ and $c(\tau+1)$. From (11) and (202) we have

$$H(\tau, \tau^\star) < I^\star e^{-(\tau^\star-\tau)} = c\tau^\star e^{-\tau^\star} e^{\tau} \to 0,$$

as $\tau^\star \to \infty$ On proceeding, in (63), to the limit $\tau^\star = \infty$, along any special sequence of numbers $\tau^\star$, we find that every limit function must satisfy the integral equation of Problem III This equation having, however, only one solution $c\tau$ for τ large, the Ergiebigkeit can have only one limit function, as $\tau^\star \to \infty$, namely $cf(\tau, \tau^\star)$.

§ 22 NARROW LIMITS FOR THE SOLUTION IN SCHWARZSCHILD'S CASE, $\gamma = 1$

In the case of uniform scattering the integral equation becomes (64), (64′), the second term on the right being (66). On setting

$$J(\tau) = I^\star u(\tau),$$

u becomes the solution of

$$u(\tau) = L(u)_\tau + \tfrac{1}{2} E_2(\tau^\star - \tau). \qquad \text{......(208)}$$

Let $v(\tau)$ denote the solution of

$$v(\tau) = L(v)_\tau + \tfrac{1}{2} E_3(\tau^\star - \tau). \qquad \text{......(209)}$$

u is then the Ergiebigkeit in the case

$$I^\star = 1, \qquad \text{......(208′)}$$

while v corresponds to the case

$$I(\tau^\star, \theta) = \cos\theta; \quad \theta < \pi/2. \qquad \text{......(209′)}$$

On introducing, for any function, the general notation

$$\overline{\Phi}(\tau) = \Phi(\tau^\star - \tau),$$

we readily find that the operator L, given by (64), satisfies

$$L(\overline{\Phi})_\tau = L(\Phi)_{\tau^\star - \tau} = \overline{L(\Phi)_\tau}. \quad \ldots (210)$$

From (207), $\gamma \equiv 1$, we have, according to (49), $n=2$,

$$1 = L(1)_\tau + \tfrac{1}{2}E_2 + \tfrac{1}{2}\overline{E}_2 \quad \ldots\ldots(211)$$

Furthermore, we find by means of (79), (81), (82), $n=1$,

$$\tau = L(t)_\tau + \frac{\tau^\star}{2}\overline{E}_2 + \tfrac{1}{2}(\overline{E}_3 - E_3) \quad \ldots\ldots (212)$$

(208), (210), (211) imply

$$u + \overline{u} - 1 = L(u + \overline{u} - 1),$$

whence, according to the uniqueness property proved in the preceding section,

$$u + \overline{u} \equiv 1 \quad \ldots(213)$$

(213) is physically obvious For $\overline{u}$ is the Ergiebigkeit when the radiation incident is zero at $\tau = \tau^\star$ and one at $\tau = 0$ $u + \overline{u}$ is therefore the Ergiebigkeit when the radiation incident is one at both faces, which yields (213)

From (209), (211), (212),

$$v - \overline{v} = \tau - \tau^\star u. \quad \ldots(214)$$

In the sequel we make use of the

LEMMA $I_1(\tau^\star, \theta) < I_2(\tau^\star, \theta)$, $\theta < \pi/2$, implies $J_1(\tau) < J_2(\tau)$ for the corresponding Ergiebigkeiten.

This follows immediately from the fact that the Ergiebigkeit is the N-solution of (64′)

Narrow limits for $u(\tau)$ can be found by comparison with the solution of Problem Ia. On setting

$$q_1 = q(\tau^\star), \quad q_2 = \int_0^\infty e^{-s} q(\tau^\star + s)\, ds, \quad \ldots\ldots(215)$$

we infer that $q_1 < q_2$ and that q_1 and q_2 increase with increasing $\tau^\star$, having both the limit q_∞ as $\tau^\star \to \infty$ In the model Ia, the intensity of the radiation coming from below is given by (100) Let us in the case $F = \frac{4}{3}$ denote this intensity by $\overline{I}$. Since $q(\tau)$ increases, we then find from (100)

$$\tau^\star + q_1 + \cos\theta < \overline{I}(\tau^\star, \theta) < \tau^\star + q_2 + \cos\theta, \quad \ldots\ldots(216)$$

for $\theta < \pi/2$ Taking $\overline{I}$ as the radiation incident at the inner face

$\tau=\tau^\star$ of model Ia, we first see that the corresponding Ergiebigkeit is precisely $f(\tau)=\tau+q(\tau)$ (according to the theory of model Ia). On the other hand, the Ergiebigkeit turns out to be

$$(\tau^\star+q_1)u+v \text{ resp } (\tau^\star+q_2)u+v,$$

when the radiation incident at $\tau=\tau^\star$ is given by the left respective right-hand term in (216) The lemma shows therefore that, for all τ between 0 and $\tau^\star$,

$$(\tau^\star+q_1)u+v<f<(\tau^\star+q_2)u+v,$$

which, for the function

$$f-\bar{f}=2\tau-\tau^\star+q-\bar{q},$$

implies the inequalities

$$f-\bar{f}>(\tau^\star+q_1)u+v-(\tau^\star+q_2)\bar{u}-\bar{v}$$

and $$f-\bar{f}<(\tau^\star+q_2)u+v-(\tau^\star+q_1)\bar{u}-\bar{v}.$$

When, here, $\bar{u}$ and $v-\bar{v}$ are expressed in terms of u by means of (213) and (214), u is found to lie between the limits

$$\frac{\tau+q-\bar{q}+q_1}{\tau^\star+q_1+q_2}<u(\tau)<\frac{\tau+q-\bar{q}+q_2}{\tau^\star+q_1+q_2}. \qquad \text{......(217)}$$

By means of the notation

$$m=m(\tau^\star)=\frac{q_1+q_2}{2}, \quad d=d(\tau^\star)=\frac{q_2-q_1}{2}, \quad \text{......(215')}$$

we can give (217) the form

$$\left|J(\tau)-I^\star\frac{\tau+q(\tau)-q(\tau^\star-\tau)+m(\tau^\star)}{\tau^\star+2m(\tau^\star)}\right|<I^\star\frac{d(\tau^\star)}{\tau^\star+2m(\tau^\star)}. \qquad \text{......(217')}$$

The use of (217) requires, of course, a numerical knowledge of the function $q(\tau)$ of Problem Ia. The limits given in (217) are much narrower than Schwarzschild's limits indicated in Theorem XVI, since $q(\tau)$ is known to lie between the narrow limits 0·577 and 0 71 The factor $d(\tau^\star)$ on the right of (217′) is about 0·05 for $\tau^\star=0$ and decreases very rapidly as $\tau^\star\to\infty$.

If, in (217), q is replaced by Milne's approximate value $\frac{2}{3}$, we obtain Milne's approximate form of J,

$$I^\star\frac{\tau+\frac{2}{3}}{\tau^\star+\frac{4}{3}},$$

while for $\tau^\star$ large, the true approximate expression is

$$I^\star \frac{\tau + q(\tau)}{\tau^\star + 2q_\infty}.$$

In the case of uniform scattering, Theorem XVII is an evident consequence of (217) or (217′)

§ 23. THE NET FLUX IN SCHWARZSCHILD'S PROBLEM

The flux constant F is not given in Schwarzschild's problem For astrophysical purposes it is, however, of importance to know $F/I^\star$ as a function of $\tau^\star$ We shall now include this function between narrow limits

First we must find the analogue of the equation (54) in Schwarzschild's model The first term on the right must, of course, be omitted, since the radiation incident at the outer face is zero Instead of this term there is the analogous term for the radiation incident at the inner face,

$$Q(\tau) = \frac{1}{\pi}\int_+ \cos\theta\, e^{-(\tau^\star-\tau)\sec\theta}\, I(\tau^\star, r)\, d\omega \quad \ldots\ldots(218)$$

Furthermore, the upper limit of integration in (54) is to be replaced by $\tau^\star$. We thus obtain Milne's flux equation

$$\tfrac{1}{2}F = \int_\tau^{\tau^\star} J(t)\, E_2(t-\tau)\, dt - \int_0^\tau J(t)\, E_2(\tau - t)\, dt + \tfrac{1}{2}Q(\tau) \quad \ldots\ldots(219)$$

For $\tau = 0$, $\tau^\star$, we have

$$\tfrac{1}{2}F = \int_0^{\tau^\star} J E_2\, dt + \tfrac{1}{2}Q(0), \quad \ldots\ldots(220)$$

$$\tfrac{1}{2}F = -\int_0^{\tau^\star} \bar{J} E_2\, dt + \tfrac{1}{2}Q(\tau^\star), \quad \ldots\ldots(220')$$

whence $$F = \int_0^{\tau^\star} (J - \bar{J})\, E_2\, dt + \tfrac{1}{2}Q(0) + \tfrac{1}{2}Q(\tau^\star). \quad \ldots\ldots(220'')$$

Let $\bar{F}$ resp F' denote the flux constant in the case (208′) resp. (209′) Taking account of (49), $n = 3$, 4, we note the following combinations in those two cases.

$$J = u, \quad Q = 2E_3(\tau^\star - \tau), \quad F = \bar{F} \quad \ldots\ldots(221)$$

and $$J = v, \quad Q = 2E_4(\tau^\star - \tau), \quad F = F' \quad \ldots\ldots(222)$$

In the present case (65) we have

$$F = I^\star \tilde{F}$$

for the net flux.

Let us, first, find the relation between F' and $\tilde{F}$ On applying (220″) to the case (222) we find, according to (214),

$$F' = \int_0^{\tau^\star} tE_2 dt - \tau^\star \int_0^{\tau^\star} uE_2 dt + \tfrac{1}{3} + E_4(\tau^\star).$$

The first integral is readily computed from (80), $n=2$, while the second one can, by means of (220), taken in the case of (221), be expressed in terms of $\tilde{F}$ Altogether, the simple relation

$$F' = \tfrac{2}{3} - \frac{\tau^\star}{2} \tilde{F} \qquad \ldots(223)$$

is obtained.

The following remark will be of use The inequality

$$I_1(\tau^\star, \theta) < I_2(\tau^\star, \theta), \quad \theta < \pi/2,$$

implies the inequality $F_1 < F_2$ for corresponding flux constants This follows from the lemma of the preceding section and from (218), (220)

We now recall the inequality (216), where the incident radiation $\bar{I}$ gives precisely the net flux $F = \frac{4}{3}$ Taking account of the above remark and of (221), (222), we find therefore

$$(\tau^\star + q_1)\tilde{F} + F' < \tfrac{4}{3} < (\tau^\star + q_2)\tilde{F} + F'$$

If this is combined with (223), we obtain simple limits for $\tilde{F} = F/I^\star$,

$$\frac{4}{3}\frac{I^\star}{\tau^\star + 2q_2} < F < \frac{4}{3}\frac{I^\star}{\tau^\star + 2q_1}. \qquad \ldots\ldots(224)$$

By means of (215′) we can write this

$$\left| \frac{1}{\tilde{F}} - \tfrac{3}{4}\tau^\star - \tfrac{3}{2}m(\tau^\star) \right| < \tfrac{3}{2}d(\tau^\star). \qquad \ldots\ldots(224')$$

This approximate expression of $1/\tilde{F}$ is, even for small $\tau^\star$, very accurate Its value for $\tau^\star = 0$ turns out to be 0·95, differing only 5 per cent. from the true value $1/\tilde{F} = 1$.

In comparison hereto, we note that the right-hand side of (224′) gives 8 per cent at $\tau^\star = 0$ For larger $\tau^\star$ the accuracy is extra-

ordinarily good. According to $m \to q_\infty$ as $\tau^\star \to \infty$, we find, in particular, from (224′),

$$\frac{1}{\bar{F}} - \tfrac{3}{4}\tau^\star \to \tfrac{3}{2}q_\infty \qquad \ldots \quad (225)$$

If q is replaced by its Milne approximation $\frac{2}{3}$, we obtain Milne's approximate formula

$$\frac{1}{\bar{F}} = \tfrac{3}{4}\tau^\star + 1.$$

The accuracy is inferred from the fact that $3m(\tau^\star)/2$ varies between the narrow limits 0.95 and 1.06.

CHAPTER IV

EXPLICIT SOLUTION OF CERTAIN INTEGRAL EQUATIONS

§ 24 THE CHARACTERISTIC EQUATION

We shall now generally treat the integral equation

$$f(x) = \int_0^\infty H(|x-y|) f(y)\, dy \qquad \ldots\ldots(226)$$

by means of Fourier integrals. The solution will be obtained in the form of explicit integral formulae.

Concerning the kernel, we suppose that, for a certain $s > 0$, $H(x) e^{sx}$ is quadratically integrable within $0 < x < \infty$. Without limitation of the generality we may suppose that

$$\int_0^\infty [H(x) e^{sx}]^2 dx < \infty; \quad s < 1 \qquad \ldots\ldots(227)$$

This hypothesis is certainly fulfilled in Milne's case. Quadratic integrability is introduced in order to apply the Plancherel Theory of Fourier Integrals

The solutions of the analogous equation

$$f(x) = \int_{-\infty}^{+\infty} H(|x-y|) f(y)\, dy \qquad \ldots\ldots(226')$$

are of much simpler form, being aggregates of exponential functions. If $u^\star$ denote an n-fold root of the 'characteristic equation'

$$1 = \kappa(u) \equiv \int_{-\infty}^{+\infty} H(|x|) e^{ux} dt, \qquad \ldots\ldots(228)$$

the function $$Q_{n-1}(x) e^{-u^\star x}$$

is easily seen to represent a solution of (226′), Q_{n-1} being an arbitrary polynomial of degree not greater than $n-1$. We must, of course, be sure that all integrals involved converge. This is, according to (227), obviously true when the real part of u lies between the limits

$$-1 < R(u^\star) < 1.$$

We should expect that the characteristic equation plays also an

important rôle in the theory of (226), and that its solutions show the same behaviour for large x as certain solutions of (226′).

We first compute the characteristic function $\kappa(u)$ in the important case, where H is of the form (156), with $\rho(s)$ increasing This includes Milne's case with $\rho=\frac{1}{2}\log s$ $\kappa(u)$ is readily seen to equal

$$\kappa(u)=\int_1^\infty\left[\frac{1}{\xi-u}+\frac{1}{\xi+u}\right]d\rho(\xi). \quad \ldots(229)$$

In order that (227) be fulfilled, we need only suppose that $\kappa(0)=2H_2(0)$ be finite. In Milne's case we have

$$\kappa(u)=\frac{1}{2u}\log\frac{1+u}{1-u}, \quad \ldots\ldots(230)$$

the logarithm being zero for $u=0$

Under the hypothesis (227), the complex function $\kappa(u)$ is certainly holomorphic in the strip

$$|R(u)|<1. \quad \ldots\ldots(231)$$

Two obvious properties are

$$\overline{\kappa(u)}=\kappa(\overline{u}), \quad \kappa(u)=\kappa(-u), \quad \ldots\ldots(232)$$

showing that $\kappa(u)$ is real on the imaginary as well as on the real axis

(227) implies absolute integrability

$$\int_0^\infty|H(x)e^{sx}|\,dx<\infty; \quad s<1, \quad \ldots\ldots(233)$$

as seen on applying Schwarz's inequality to (233), the integrand being

$$\left(H(x)e^{\frac{s+1}{2}x}\right)e^{\frac{s-1}{2}x}.$$

By simple substitution $x=x'+\pi/t$, $\kappa(u)=\kappa(s+it)$ can be written in the form

$$2\kappa(u)=\int_{-\infty}^{+\infty}\left[H(|x|)e^{sx}-H\left(\left|x+\frac{\pi}{t}\right|\right)e^{s\left(x+\frac{\pi}{t}\right)}\right]e^{itx}dx,$$

whence

$$2|\kappa(u)|<\int_{-\infty}^{+\infty}\left|H\left(\left|x+\frac{\pi}{t}\right|\right)-H(|x|)\right|e^{s|x|}dx$$

$$+\left|e^{-s\frac{\pi}{t}}-1\right|\int_{-\infty}^{+\infty}|H(|x|)|\,e^{s|x|}dx.$$

If H is continuous, this inequality shows that

$$\kappa(s+it) \to 0 \quad \text{as} \quad |t| \to \infty \qquad \ldots\ldots(234)$$

holds uniformly in every partial strip $|s| \leqq a < 1$ Theorems of Lebesgue, however, show that this still holds under mere measurability of H As a consequence we note that the characteristic equation can have only a finite number of roots within every partial interval $|s| \leqq a < 1$

More can only be said about the characteristic roots when H satisfies special hypotheses In the important case, for instance, where H is positive, $\kappa(u)$ is found to increase from $\kappa(0)$ to $\kappa(1) \leqq \infty$ when u goes from 0 to 1 along the real axis (or from 0 to -1) On the imaginary axis $\kappa(u)$ is still real and less than $\kappa(0)$ in absolute value The characteristic equation has therefore the double root $u=0$ when

$$\kappa(0)=1, \qquad \ldots\ldots(235)$$

two real roots of opposite sign, however, when

$$\kappa(0)<1<\kappa(1), \qquad \ldots\ldots(235')$$

and at least two imaginary roots when

$$\kappa(0)>1 \qquad \ldots(235'')$$

Besides these it could very well have other complex roots in the strip $|s|<1$.

Let us, now, suppose H to be of the form (156) with increasing ρ In this case, there are, within $|Q(u)|<1$, no other roots than $u=0$ in the case of (235), than two opposite real roots in case of (235′) and than two opposite imaginary roots in the case of (235″) In Milne's case, for instance, all characteristic roots are thus exhausted by the double root $u=0$

$$\kappa(u)-\kappa(\bar{u})=8ist\int_1^\infty \frac{\xi\,d\rho(\xi)}{|\xi^2-u^2|^2}; \quad u=s+it,$$

which, according to (232), shows that κ can only be real on the real or imaginary axis Furthermore, the equation

$$\kappa'(it)=4it\int_1^\infty \frac{\xi\,d\rho(\xi)}{(\xi^2+t^2)^2}$$

proves that κ decreases when u goes from 0 to $+i\infty$ or to $-i\infty$ along the imaginary axis The statement about the roots of $\kappa=1$

is an immediate consequence of these facts It may be added that, under the condition (172), i e $\kappa(1)=\infty$, the right-hand inequality of (135′) is automatically satisfied

§ 25. THE INTEGRAL FORMULAE FOR THE SOLUTION OF (226)

We shall now construct all solutions of (226) that satisfy the condition

$$f(x)=O(e^{\alpha x}), \quad \alpha<1, \qquad \ldots (236)$$

for x large, α being an arbitrarily fixed number less than one

Let

$$u_1, u_2, \ldots, u_{2n}$$

denote the complete set of characteristic roots within the strip $|s| \leqq \alpha$. If necessary we enlarge α a little such that no roots lie on the boundaries $|s|=\alpha$. We set

$$P(u)=(u-u_1)(u-u_2)\ldots(u-u_{2n}), \qquad \ldots (237)$$

$P(u)$ being even and real on the real axis, and

$$\tau(u)=\frac{(u^2-1)^n}{P(u)}\{1-\kappa(u)\}, \qquad \ldots (238)$$

τ being holomorphic and free of zeros in the strip $|s| \leqq \alpha$. Moreover, $\tau(u)$ is even and real for real u We want $\tau(u)$ in the form $\tau(u)=\tau_+(u)/\tau_-(u)$, in such a way that $\tau_+(u)$ is holomorphic in the half-plane $s \geqq -\alpha$ and that $\tau_+(s+it)\to 1$ as $s\to+\infty$, while $\tau_-(s+it)$ is holomorphic for $s \leqq \alpha$ and satisfies $\tau_-(s+it)\to 1$ as $s\to-\infty$. This obviously unique representation is, according to Cauchy's formula, explicitly furnished by

$$\tau_+(u)=\exp\left[-\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}\frac{\log\tau(v)}{v-u}\,dv\right], \ldots (239)$$

$$\tau_-(u)=\exp\left[-\frac{1}{2\pi i}\int_{+\beta-i\infty}^{+\beta+i\infty}\frac{\log\tau(v)}{v-u}\,dv\right], \quad \ldots (239')$$

where $\beta>\alpha$ is chosen such that the strip $|s| \leqq \beta$ contains no new roots The logarithm $\log\tau(s+it)$ is the one that vanishes as $t\to+\infty$ Observing that $\tau(u)$ is never zero, and real on the imaginary axis, and that $\tau(s\pm i\infty)=1$, we see that $\tau(u)$ is necessarily positive along that axis $\log\tau(s+it)$ vanishes therefore too,

when $t \to -\infty$ We shall later see that the integrals (239) and (239′) converge absolutely

On setting

$$\sigma_+(u) = (u+1)^{-n}\tau_+(u), \quad \sigma_-(u) = (u-1)^n \tau_-(u), \qquad \ldots (240)$$

we state the

THEOREM XVIII The integral equation (226) has precisely n linearly independent solutions satisfying (236). They can be expressed by means of the formulae

$$\phi(u) = Q_{n-1}(u)\frac{\sigma_-(u)}{P(u)}, \qquad \ldots\ldots(241)$$

Q_{n-1} being an arbitrary polynomial of degree less than n, and

$$f(x) = \frac{1}{\sqrt{2\pi i}}\int_{s-i\infty}^{s+i\infty} \phi(u)\,e^{-ux}\,du, \quad s = -\frac{\alpha+\beta}{2}. \qquad \ldots\ldots(242)$$

COROLLARY The solutions have, for large x, the form

$$f(x) = \Sigma Q^\star(x)\,e^{-u^\star x} + O(e^{-\alpha x}),$$

$u^\star$ being a characteristic root within $|s|<\sigma$, and $Q^\star$ being a polynomial of degree less than the multiplicity of $u^\star$.

It must be emphasized that, while (226′) has precisely $2n$ solutions with the property (236), (226) has only half the number of solutions.

The abscissa of integration s in (242) can be moved without change of the integral, provided that no pole of $\phi(u) = \phi(s+it)$, i e no characteristic root is met by it. $f(x)$ vanishes for $x<0$.

Application to Milne's case. Here we have, according to (30),

$$\tau(u) = \frac{u^2-1}{u^2}\left(1 - \frac{1}{2u}\log\frac{1+u}{1-u}\right). \qquad \ldots\ldots(243)$$

$\tau_-(u)$ is given by (239′), while

$$\phi(u) = c\,\frac{u-1}{u^2}\,\tau_-(u), \qquad \ldots\ldots(244)$$

c being an arbitrary constant of proportionality. If (242) is considered as a Fourier integral, we find by the Fourier theorem

$$\phi(s+it) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\,e^{(s+it)x}\,dx.$$

This represents $\phi(u)$ for all $s<0$, since the integral converges and is holomorphic in the left half-plane We can obtain herefrom an explicit formula for the law of darkening in Problem Ia From the last formula, $t=0$, and from (188) and (189) we get

$$I(0,\theta)=\tfrac{3}{4}F\sqrt{2\pi}\sec\theta\,\phi(-\sec\theta) \qquad . \quad . \quad (245)$$

The proportionality constant c in (243) remains to be determined such that the corresponding solution $f(x)$ is $\sim x$ for x large From (101), $I\left(0,\frac{\pi}{2}\right)=\frac{\sqrt{3}}{4}F$, which gives, together with (245),

$$\frac{1}{\sqrt{6\pi}}=-\lim_{s=-\infty} s\phi(s)$$

On the other hand, from (239′) and from (244), we find that this limit equals c. We therefore obtain the law of darkening in the explicit form

$$I(0,\theta)=\frac{\sqrt{3}}{4}F(1+\cos\theta)\,\tau_{-}(-\sec\theta), \quad \ldots\ldots(246)$$

τ_{-} being determined by means of (239′) and (243). The integral in (239′) could be found by numerical integration

Plancherel's theorem on Fourier integrals. It is convenient to state here the main theorems of the modern theory of Fourier integrals as far as we need them in the next sections A complex valued function $a(x)$ of the real variable x is called quadratically summable (q.s.) over $-\infty<x<+\infty$ if

$$\int_{-\infty}^{+\infty}|a(x)|^2\,dx$$

is finite. According to Plancherel, the Fourier transform

$$A(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{+itx}\,a(x)\,dx$$

exists then in the sense of mean convergence. $A(t)$ is again q.s. over $-\infty<t<+\infty$, and $\bar{a}(x)$ is conversely the Fourier transform of $\bar{A}(x)$,

$$a(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-ixt}A(t)\,dt.$$

Two q s functions a, b and their Fourier transforms A, B satisfy the Parseval relation

$$\int_{-\infty}^{+\infty} a(x)\,\overline{b(x)}\,dx = \int_{-\infty}^{+\infty} A(x)\,\overline{B(x)}\,dx$$

In particular, we have

$$\int_{-\infty}^{+\infty} |a(x)|^2 dx = \int_{-\infty}^{+\infty} |A(x)|^2 dx$$

The integral (242) is to be understood in the same sense of mean convergence

§ 26 PRELIMINARIES

Properties of the function $\tau(u)$. We know already that $\tau(u)$ is holomorphic and different from zero within $|s| \leqq \beta$ $(\beta > \alpha)$ Moreover, we know that $\tau(u)$ converges to one as u goes to infinite within that strip $\tau(u)$ being real on the real and imaginary axes, we infer its positivity on these axes. The logarithm of $\tau(u)$ considered in (139) and (139′) is therefore real on both axes and tends to zero as u goes to infinity within the strip $\log\tau(u)$ is, furthermore, an even function of u

According to (228), $\kappa(s+it)/\sqrt{2\pi}$ is the Fourier transform of the q s function $e^{sx}H(|x|)$, and represents, according to Plancherel's theorem, a q s. function of t, $|s|<1$. (237) and (238) show therefore that $\log\tau(s+it)$ is a q s function of t too. We can, from this fact, infer the absolute convergence of the integrals (239) and (239′) By means of Schwarz's inequality we find

$$4\pi^2 |\log\tau_-(u)|^2 \leqq \int_{\beta-i\infty}^{\beta+i\infty} |\log\tau(u)|^2 |dv| \int_{\beta-i\infty}^{\beta+i\infty} \frac{|dv|}{|u-v|^2}.$$

This shows, moreover, that $\log\tau_-(u)$, being holomorphic in $s<\beta$, is bounded in every partial half-plane $s \leqq \beta' < \beta$ Similar facts are true for $\log\tau_+(u)$ within the half-plane $s > -\beta$

The functions σ_+ and σ_- introduced by (240) have therefore the properties

$$m|u|^{-n} < |\sigma_+(u)| < M|u|^{-n}; \quad s \geqq -\frac{\alpha+\beta}{2}, \ \ldots(247)$$

and
$$m\,|u|^n < |\sigma_-(u)| < M\,|u|^n, \quad s \leqq \frac{\alpha+\beta}{2}, \dots \quad (247')$$
with suitable positive constants m, M

We mention, finally, that the equation $\tau = \tau_+/\tau_-$ can, according to (238) and (240), be written
$$1-\kappa(u) = P(u)\frac{\sigma_+(u)}{\sigma_-(u)}. \qquad .\,.(248)$$

§ 27. PROOF OF THEOREM XVIII

We write (246) in the form
$$g(x) = f(x) - \int_{-\infty}^{+\infty} H(|x-y|) f(y)\,dy, \qquad \dots (249)$$
where
$$f(x) = 0, \quad x < 0, \quad g(x) = 0, \quad x > 0 \qquad . \quad (250)$$
For $x < 0$, $g(x)$ is understood to be defined by the right-hand side of (249). We now set
$$\phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,e^{ux}\,dx, \quad \gamma(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(x)\,e^{ux}\,dx, \qquad \dots . (251)$$
in other words, we consider the Fourier transforms
$$\phi(s+it), \quad \gamma(s+it)$$
(regarded as functions of t) of $f(x)\,e^{sx}$ and $g(x)\,e^{sx}$.

Since $f(x)$ is supposed to satisfy (236), we may state the obvious

LEMMA 1. $\phi(u)$ is holomorphic in the half-plane $s < -\alpha$, and bounded in every partial half-plane, for instance, in $s \leqq -(\alpha+\beta)/2$

From (249), $x < 0$, and from (250),
$$|g(x)| \leqq \int_0^{+\infty} |H(|x-y|)|\,|f(y)|\,dy$$
$$< \text{const} \int_0^{+\infty} |H(|x-y|)|\,e^{\alpha y}\,dy.$$
For any λ between α and 1, we find
$$\int_0^{+\infty} |H(|x-y|)|\,e^{\alpha y}\,dy < \int_{-\infty}^{+\infty} |H(|x-y|)|\,e^{\lambda y}\,dy$$
$$= e^{\lambda x}\int_{-\infty}^{+\infty} |H|\,e^{\lambda t}\,dt,$$
whence, for $x < 0$, $\qquad g(x) = O(e^{-\lambda|x|})$,

λ being an arbitrary number less than one Consequently, the Fourier transform has the following property

LEMMA 2 $\gamma(u)$ is holomorphic in the half-plane $s > -1$ and bounded in every partial half-plane, for instance, in $s \geqq -(\alpha+\beta)/2$

It is the chief point of the following considerations that the regularity half-planes of ϕ and u have a common ordinate, $s = -(\alpha+\beta)/2$. Along such a common ordinate, we have, from (249),

$$\gamma(u) = \phi(u) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ux} dx \int_{-\infty}^{+\infty} H(|x-y|) f(y)\, dy$$

$$= \phi(u) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, dy \int_{-\infty}^{+\infty} H(|x-y|)\, e^{ux}\, dx$$

$$= \phi(u) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{uy}\, dy \int_{-\infty}^{+\infty} H(|t|)\, e^{ut}\, dt,$$

hence $$\gamma(u) = \phi(u)\{1 - K(u)\}; \quad s = -(\alpha+\beta)/2 \quad \ldots (252)$$

It is permitted to interchange the order of integration along the common ordinate, since the double integral converges absolutely for $s = -(\alpha+\beta)/2$ The identity (248) allows us to write (252) in the form

$$\frac{\gamma(u)}{\sigma_+(u)} = \frac{\phi(u)}{\sigma_-(u)} P(u), \qquad \ldots\ldots(253)$$

$s = -(\alpha+\beta)/2$. Here, we know that the left-hand side is holomorphic for $s \geqq -(\alpha+\beta)/2$, while the right-hand side is holomorphic within $s \leqq -(\alpha+\beta)/2$. Both sides thus define an entire function of u. $P(u)$ being a polynomial of degree $2n$, we infer from (247) and (247′) that this entire function is at most of the order of u^n for $|u|$ large. According to a well-known theorem of the theory of complex functions it is therefore a polynomial $Q(u)$ of degree not greater than n, whence

$$\phi(u) = \sigma_-(u) \frac{Q(u)}{P(u)}.$$

Q can, according to (247′), not be of degree n since $\phi(s+it)$, being the Fourier transform of a q.s function, is again a q.s. function of t It is thus proved that $\phi(u)$ has the form (241).

We must not forget to prove that (241) and (242) actually

represent solutions of (228) We now start with (241) and study the functions $f(x)$ defined by (242) (241) and (247′) show that $\phi(u)=O(|u|^{-1})$ for $|u|$ large, in particular, that $\phi(s+it)$ is a q.s function of t Taking account of Lemma 1 and of Cauchy's theorem, we see that the abscissa of integration in (242) may be moved to the left without change of the integral For any $s\leqq(\alpha+\beta)/2$, $\phi(s+it)$ is, conversely, the Fourier transform of $f(x)e^{sx}$. Hence and from Parseval's formula, $a=b$, $A=B$,

$$\int_{-\infty}^{+\infty}|f(x)|^2e^{2sx}\,dx=\int_{-\infty}^{+\infty}|\phi(s+it)|^2\,dt.$$

The right-hand side being bounded for $s\leqq-(\alpha+\beta)/2$, we infer, on proceeding to the limit $s\to-\infty$, that $f(x)$ vanishes for, at least, almost all negative x (in the sense of Lebesgue) Otherwise the left-hand side would increase indefinitely as $s\to-\infty$

Let us, now, set
$$\gamma(u)=\sigma_+(u)\,Q_{n-1}(u) \qquad\qquad .(254)$$
and
$$g(x)=\frac{1}{\sqrt{2\pi}\,i}\int_{s-i\infty}^{s+i\infty}\gamma(u)\,e^{-ux}\,du,\quad s=-\frac{\alpha+\beta}{2} \qquad ...\,(255)$$

According to (247), $\gamma(u)$ is regular and $O(|u|^{-1})$ in $s\geqq-(\alpha+\beta)/2$ The abscissa of integration can, in (255), be placed arbitrarily far to the right, whence we infer that $g(x)$ vanishes for almost all positive x.

It remains to prove that $f(x)$ and $g(x)$, found from (242) and (255), satisfy (249) In other words, we have to show that, backwards, the relation (252), being satisfied by our ϕ and γ, implies (249) We note that the following functions A, B are the Fourier transforms of a, b,

$$a(y)=e^{sy}f(y),\quad A(t)=\phi(s+it)=\phi(u),$$
$$b(y)=e^{s(x-y)}H(|x-y|),\quad B(t)=\frac{e^{itx}}{\sqrt{2\pi}}\overline{\kappa(u)},$$

always on the ordinate $s=-(\alpha+\beta)/2$. The Parseval relation $(a,\bar b)=(A,\bar B)$ gives then, $s=-(\alpha+\beta)/2$,

$$e^{sx}\int_0^\infty H(|x-y|)f(y)\,dy=\frac{1}{\sqrt{2\pi}\,i}\int_{s-i\infty}^{s+i\infty}\phi(u)\,\kappa(u)\,e^{-ux}\,du. \qquad ......(256)$$

On combining this with (242) and (255), and on taking account of (252), we see that (249) and (250), i e that (226) is satisfied The right-hand integral in (256) converges absolutely because ϕ and κ are both q s functions of t Both sides in (256) represent, therefore, bounded functions of x This implies, according to (226),

$$f(x) = O\left(e^{\frac{\alpha+\beta}{2}x}\right).$$

The coefficient of the exponent is, here, greater than α We remember, however, that σ was subjected to the only restriction that no characteristic roots should lie on $s = \pm\sigma$. Our formulae would, therefore, not change if a slightly smaller σ be employed and if β also be taken smaller than the original σ. There is thus no difficulty in establishing (236).

§ 28 PROOF OF THE ASYMPTOTIC FORM OF THE SOLUTIONS

According to (241), $\phi(u)$ can have its poles only among the characteristic roots It has, however, at least $(n+1)$ poles within $|s| < \alpha$ When, in (242), the abscissa of integration $s = -(\alpha+\beta)/2$ is moved to $s = (\alpha+\beta)/2$, the change of the integral amounts to $-\sqrt{2\pi}$ times the sum of the residues of the integrand, contained in the strip $|s| < \sigma$. Near a characteristic root $u^\star$ of order k, the integrand is of the form

$$e^{-xu}\ h(u)\, Q_{n-1}(u)\, \frac{e^{-x(u-u^\star)}}{(u-u^\star)^k}; \quad h(u^\star) \neq 0.$$

The residue at $u = u^\star$ has, therefore, the form

$$Q^\star(x)\, e^{-u^\star x},$$

$Q^\star$ being a polynomial of degree less than k. In the case $Q_{n-1}(u^\star) \neq 0$ the degree equals precisely $k-1$. Altogether we obtain

$$f(x) = f^\star(x) + r(x),$$

where $f^\star(x) = O(e^{\alpha|x|})$ is a solution of (226′), and where

$$r(x) = \frac{1}{\sqrt{2\pi}\,i}\int \phi(u)\, e^{-ux}\, du,$$

the abscissa of integration now being $s=(\alpha+\beta)/2$ For $x>0$, (226′) can be written

$$f^{\star}(x)-\Lambda\,(f^{\star})_x=\int_{-\infty}^{0} H\,(x+|\,y\,|)f^{\star}(y)\,dy,$$

Λ being the operator from 0 to $+\infty$, given by (226). This implies, according to $f^{\star}=O\,(e^{\alpha|x|})$ and according to (233),

$$f^{\star}(x)-\Lambda\,(f^{\star})_x=O\,(e^{-\alpha x}),$$

for $x>0$. The same holds, according to $f=\Lambda\,(f)$, for the function $r\,(x)$,

$$r\,(x)-\Lambda\,(r)_x=O\,(e^{-\alpha x}). \qquad \ldots\ .(257)$$

On the other hand, $r\,(x)\,e^{sx}$, $s=(\alpha+\beta)/2$, is q s , being the Fourier transform of $\phi\,(u)$, along $s=(\alpha+\beta)/2$. On applying Schwarz's inequality to $\Lambda\,(r)_x$, the integrand being written

$$[H\,(|\,x-y\,|)\,e^{-sy}]\,[r\,(y)\,e^{sy}],$$

we infer, therefore, that $r\,(x)=O\,(e^{-\alpha x})$, which completes the proof of the corollary.

§ 29. NEW PROOF OF THEOREM X

The essential hypothesis, for the validity of Theorem X, was the positivity of the kernel, $H>0$, and (119), which can be written

$$\kappa\,(0)=2H_2\,(0)=1, \qquad \ldots\ldots(258)$$

i e. $u=0$ is a characteristic root, being at least double. According to $H>0$, we have, however,

$$\kappa''\,(0)=2\int_0^{\infty} t^2H\,(t)\,dt=4H_1\,(0)>0, \qquad \ldots.\ .(259)$$

which shows that $u=0$ is precisely double. There are no other real roots and, according to $|\,\kappa\,(it)\,|<\kappa\,(0)$, $t\neq 0$, no imaginary roots. Other complex roots could, however, well exist. The polynomial $P\,(u)$ of (237) can thus be written in the form

$$P\,(u)=u^2R\,(u)\,S\,(u),$$

where

$$R\,(u)=(u-u_1)\ldots(u-u_{n-1}),$$

$$S\,(u)=(u+u_1)\ \ldots(u+u_{n-1}).$$

On making, in (241), the particular choice

$$Q_{n-1}(u) = cR(u),$$

we find
$$\phi(u) = c\frac{\sigma_-(u)}{u^2 S(u)}, \qquad \ldots(260)$$

and, for suitable c,
$$f(x) = x + q_\infty + o(1) \qquad \ldots(261)$$

(concerning the notation see Theorem VI), since the roots of $S(u)=0$ all lie in the right half-plane $s>0$, giving rise to residues that vanish exponentially as $x \to 0$ At the same time it is seen that no other choice for Q_{n-1} leads to solutions of the form (261)

From (239′), (240) and (260),

$$u\phi(u) \to c, \quad s \to \infty, \qquad \ldots(262)$$

and
$$u^2\phi(u) \to c\frac{\sigma_-(0)}{S(0)}; \quad u \to 0 \qquad \ldots(263)$$

On the other hand, the first formula (251) is a consequence of (242) and holds, here, for all u with $s<0$,

$$\phi(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\, e^{sx}\, dx. \qquad \ldots(264)$$

In order to determine c, we observe that (264) implies

$$s\phi(s) \to -\frac{f(0)}{\sqrt{2\pi}}; \quad s \to -\infty, \qquad \ldots(265)$$

while (261) and (264) imply

$$s^2\phi(s) \to \frac{1}{\sqrt{2\pi}}; \quad s \to 0. \qquad \ldots(266)$$

By combination of (262), (265) and (263), (266),

$$f_0 = \frac{S(0)}{\sigma_-(0)}. \qquad \ldots(267)$$

$\sigma_-(0)$ can be found from (239′) and (240),

$$\sigma_-(0) = (-1)^n \exp\left[-\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\log\tau(v)}{v}\,dv\right]. \qquad \ldots(268)$$

The abscissa of integration can, here, be moved to the left and deformed into another path, consisting of the parts $(-i\infty, -i\rho)$ and $(i\rho, i\infty)$ of the imaginary axis, and of the right half of the

circle $|v|=\rho$ The integrals along the first two pieces cancel each other, $v^{-1}\log\tau(v)$ being an odd function The expression inside of the parenthesis, in (268), equals therefore

$$-\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log\tau(\rho e^{i\phi})\,d\phi,$$

whence, on proceeding to the limit $\rho=0$, the value

$$-\tfrac{1}{2}\log\tau(0)$$

follows. From (238),

$$\tau(0)=(-1)^{n+1}\frac{\kappa''(0)}{2R(0)\,S(0)}=\frac{2H_4(0)}{S^2(0)},$$

hence

$$\sigma_-(0)=(-1)^n S(0)/\sqrt{2H_4(0)},$$

and according to (267),

$$f_0=(-1)^{n-1}\sqrt{2H_4(0)}.$$

It is to be noted that $n-1$ is always even, since the conjugate of a characteristic root is again such a root.

Remark on q_∞ Computations which may be left to the reader lead to the following value of q_∞ in (261),

$$q_\infty=\frac{1}{\pi}\int_0^\infty\left\{\frac{2}{t}+i\frac{\kappa'(it)}{1-\kappa(it)}\right\}dt.$$

In Milne's case this can be transformed into

$$q_\infty=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left\{\frac{3}{\sin^2\phi}-\frac{\tan\phi}{\phi-\tan\phi}\right\}d\phi.$$

The evaluation gives 0·710.

CHAPTER V

OTHER PROBLEMS OF RADIATIVE EQUILIBRIUM

§ 30 PURE ABSORPTION NON GRAY MATERIAL

In this section, a few principal remarks will be made concerning the fundamental integral equation in the case of purely absorbing material with an absorption coefficient that varies with the wave-length. In the case of thermodynamical equilibrium,

$$\eta_\nu = \alpha_\nu B_\nu ,$$

where B_ν is a given function of B,

$$B_\nu = B_\nu(B), \quad B \equiv \int_0^\infty B_\nu(B)\, d\nu, \qquad \ldots\ldots(269)$$

with the following properties,

$$\frac{dB_\nu}{dB} > 0, \quad B_\nu(0) = 0, \quad B_\nu \to \infty, \quad B \to \infty \quad \ldots\ldots(269')$$

Only these qualitative properties will be used in the sequel In order to simplify the considerations we suppose that the absorption coefficient be of the form

$$\alpha_\nu(P) = \beta(\nu)\,\alpha(P). \qquad \ldots\ldots (270)$$

The material is again supposed to be stratified in parallel planes. The variable τ defined by

$$\tau = \int_{-\infty}^{x} \rho(x)\, \alpha(x)\, dx$$

is no more the optical depth though being convenient for the mathematical discussion. We assume the τ-thickness of the slab to be finite, $\tau < \tau^\star$. The true optical depth of a point x, corresponding to ν, is

$$\tau_\nu = \int_{-\infty}^{x} \rho\alpha_\nu\, dx = \beta_\nu \tau,$$

writing briefly β_ν instead of $\beta(\nu)$.

The equation of transfer is

$$\frac{\cos\theta}{\beta_\nu}\frac{\partial I_\nu}{\partial\tau}=I_\nu-B_\nu, \qquad \ldots. \ (271)$$

while the equation of radiative equilibrium (19) becomes

$$\int_0^\infty \beta_\nu B_\nu d\nu=\int_0^\infty \beta_\nu d\nu \frac{1}{4\pi}\int I_\nu d\omega. \qquad ..(272)$$

The net flux πF of the radiation of all frequencies is, of course, constant. When (271) is multiplied with $\cos\theta$ and integrated with respect to $\imath$ and ν, we find

$$K=F\tau+\text{const.},$$

$$K=\int_0^\infty K_\nu d_\nu, \quad K_\nu=\frac{1}{\pi\beta_\nu}\int I_\nu \cos^2\theta d\omega, \ \ldots \ (273)$$

which represents a generalization of Eddington's relation (32), K having, however, no immediate physical meaning in this case.

Problem V Find $I_\nu(\tau,\theta)$ and $B_\nu(\tau)$ from (269), (271) and (272) when the radiation incident at the surface $\tau=0$ is zero, while the radiation incident at the inner face $\tau=\tau^\star$ has a prescribed value $I_\nu{}^\star$, independent† of the direction ($\theta<\pi/2$)

We shall first find the integral equation for $B(\tau)$. On integrating (271),

$$I_\nu(\tau,\theta)=\begin{cases} I_\nu{}^\star e^{-\beta_\nu\sec\theta(\tau^\star-\tau)}+\beta_\nu\sec\theta\int_\tau^{\tau^\star} e^{-\beta_\nu\sec\theta(t-\tau)}B_\nu dt, & \theta<\frac{\pi}{2}, \\ \beta_\nu\,|\sec\theta\,|\int_0^\tau e^{-\beta_\nu\,|\sec\theta|(\tau-t)}B_\nu dt, & \theta>\frac{\pi}{2}, \end{cases} \qquad \ldots\ldots(274)$$

and on integrating, now, through all directions,

$$\frac{1}{4\pi}\int I_\nu d\omega=\frac{\beta_\nu}{2}\int_0^{\tau^\star} E_1(\beta_\nu\,|\,\tau-t\,|)\,B_\nu\{B(t)\}\,dt+\frac{I_\nu{}^\star}{2}E_2\{\beta_\nu(\tau^\star-\tau)\}. \qquad (275)$$

We introduce the brief notation

$$\Phi(x,B)=\int_0^\infty \beta_\nu{}^2 E_1(\beta_\nu x)\,B_\nu(B)\,d\nu \qquad \ldots..(276)$$

and

$$\Psi(B)=\int_0^\infty \beta_\nu B_\nu(B)\,d\nu. \qquad \ldots..(277)$$

† This is unessential for the subsequent considerations.

When the coefficient of absorption is a known functio
and frequency, Φ and Ψ are to be considered definite fı
their arguments. From (272) and (275), we obtain 1
mental integral equation

$$\Psi\{B(\tau)\} = \tfrac{1}{2}\int_0^{\tau^\star} \Phi\{|\tau - t|, B(t)\}\,dt + \tfrac{1}{2}G(\tau^\star - \tau$$

where
$$G(x) = \int_0^\infty \beta_\nu I_\nu{}^\star E_2(\beta_\nu x)\,d\nu, \quad \ldots$$

for the determination of $B(\tau)$ Once (278) is solved v
to determine $B_\nu = B_\nu\{B(\tau)\}$ and, from (274), $I_\nu(\tau, 0)$

§ 31. DISCUSSION OF THE NON LINEA INTEGRAL EQUATION

The difficulties connected with the Problem V, pı
already in § 4, compel us to content ourselves with soı
tive remarks on the solution of (278). Again, it is con
introduce a brief symbol

$$O(B)_\tau \equiv \tfrac{1}{2}\int_0^{\tau^\star} \Phi\{|\tau - t|, B(t)\}\,dt \quad \ldots$$

for the non-linear integral operator within (278). (27
then
$$\Psi\{B(\tau)\} = O(B)_\tau + \tfrac{1}{2}G(\tau^\star - \tau).$$

In the special case of gray material, $\beta_\nu \equiv 1$, O become
operator L of Schwarzschild, while $G(x)$ simply becom

The operator O has a fundamental property whi
called 'monotonity' and which corresponds to positi
linear case Since (276) represents, according to (2
creasing function of B,

$$B_1(\tau) \leqq B_2(\tau), \quad B_1 \not\equiv B_2$$

implies
$$O(B_1)_\tau < O(B_2)_\tau$$

everywhere We also note that the function $\Psi(B)$ in
increasing function of B.

Theorem XIX. Problem V has a solution. The int
tion (278) can be solved by successive approximation

Proof We first study $O(B)_\tau$ in the case of a constant $B(\tau)$ For any ν, $B_\nu(\tau)$ is then constant too We remember that $O(B)$ is obtained from the first term on the right of (275) by multiplication with β_ν and by integration through the spectrum This term equals, now,

$$\frac{B_\nu}{2}\{2-E_2(\beta_\nu\tau)-E_2[\beta_\nu(\tau^\star-\tau)]\},$$

whence
$$O(B)_\tau=\Psi(B)-\tfrac{1}{2}\int_0^\infty \beta_\nu B_\nu E_2(\beta_\nu\tau)\,d\nu$$
$$-\tfrac{1}{2}\int_0^\infty \beta_\nu B_\nu E_2[\beta_\nu(\tau^\star-\tau)]\,d\nu, \quad \ldots(281)$$

in the case of a constant B. We might, together with (276), $\Phi=\Phi_1$, introduce the functions

$$\Phi_n(x,B)=\int_0^\infty \beta_\nu^{3-n}E_n(\beta_\nu x)\,B_\nu(B)\,d\nu \quad . \quad (276')$$

(281) then takes the form

$$\Psi(B)=O(B)_\tau+\tfrac{1}{2}\Phi_2(\tau,B)+\tfrac{1}{2}\Phi_2(\tau^\star-\tau,B) \quad \ldots .(281')$$

From (269′) we obtain the following property of the functions Φ_n, in particular of Φ_2,

$$\Phi_2(x,B)\to\infty, \quad B\to\infty$$

This property shows that, for all large enough constants $\bar{B}$,

$$\Phi_2(x,\bar{B})>G(x),$$

$0<x<\tau^\star$, or, according to (281′), $B=\bar{B}$, that the inequality

$$\Psi(\bar{B})>O(\bar{B})_\tau+\tfrac{1}{2}G(\tau^\star-\tau), \quad \ldots ..(282)$$

$0<\tau<\tau^\star$, holds for all sufficiently large constants $\bar{B}$ This proves the convergence of the successive approximations in (278). We set

$$\Psi\{B_{n+1}(\tau)\}=O(B_n)_\tau+\tfrac{1}{2}G(\tau^\star-\tau), \quad \ldots . (283)$$

$B_0(\tau)\equiv 0$ $\Psi(B)$ being an increasing function of B, (283) uniquely defines a series of functions $B_n(\tau)$, $n=0, 1, 2, \ldots$ We obviously have

$$B_1(\tau)>B_0(\tau).$$

Supposing the inequality $B_n(\tau)>B_{n-1}(\tau)$ to be true for one particular n, we find from (283), according to the monotonity of the operator O,

$$\Psi\{B_{n+1}(\tau)\}>O(B_{n-1})_\tau+\tfrac{1}{2}G(\tau^\star-\tau)=\Psi(B_n(\tau)),$$

whence $B_{n+1}(\tau) > B_n(\tau)$ Induction shows thus the general validity of this inequality On the other hand, from $B_0(\tau) < \bar{B}$,

$$\Psi\{B_1(\tau)\} < O(\bar{B})_\tau + \tfrac{1}{2}G(\tau^\star - \tau),$$

and according to (282), $B_1(\tau) < \bar{B}$. Continuing in this way we generally find $B_n(\tau) < \bar{B}$. The limit function

$$B(\tau) = \lim_{n=\infty} B_n(\tau)$$

is as before seen to satisfy the integral equation (278), q e.d. The proof of the uniqueness may be omitted.

§ 32 REMARKS ON THE CASE OF INFINITE OPTICAL DEPTH

Theorem I is probably generally true We shall indicate the proof in the present case

According to $I_\nu \geqq 0$, we find as previously

$$I_\nu(\tau, r) = e^{\beta_\nu \sec\theta}\, i_\nu(r) + \beta_\nu \sec\theta \int_\tau^\infty e^{-\beta_\nu \sec\theta (t-\tau)} B_\nu\, dt, \quad \ldots\ldots (284)$$

$i_\nu(r) \geqq 0$, for the radiation from below, $\theta < \pi/2$ According to $B_\nu \geqq 0$,

$$I_\nu(\tau, r) \geqq e^{\beta_\nu \tau} i_\nu(r); \quad \theta < \pi/2,$$

and according to (273),

$$K_\nu(\tau) > \frac{e^{\beta_\nu \tau}}{\pi \beta_\nu} \int_+ i_\nu(r) \cos^2\theta\, d\omega, \qquad \ldots\ldots (285)$$

thus yielding

$$K(\tau) > \int_+ j(r, \tau) \cos^2\theta\, d\omega, \quad j = \frac{1}{\pi} \int_0^\infty \frac{e^{\beta_\nu \tau}}{\beta_\nu} i_\nu(r)\, d\nu.$$

This is readily seen to imply $i_\nu(r) = 0$ for almost all (ν, r), because, otherwise, $K(\tau)$ would increase exponentially with increasing τ, in contradiction to (273).

Problem VI Find B and I_ν, the incident radiation being zero, and the flux constant $F \geqq 0$ being given.

The integral equation of this problem becomes, according to the notation introduced in § 31,

$$\Psi\{B(\tau)\} = \tfrac{1}{2} \int_0^\infty \Phi\{|\tau - t|, B(t)\}\, dt. \qquad \ldots\ldots (286)$$

It is natural to conjecture that this equation has a one-parameter family of solutions $B(\tau, c)$. In the simplest case of gray material, $\beta_\nu = 1$, (286) becomes Milne's linear equation. In the present general case, the connection between B and F is no longer linear. In analogy to the linear case, we could also insert (274), $\tau^\star = \infty$, directly into the flux integral, whence, according to (276′),

$$F = 2\int_\tau^\infty \Phi_2\{t-\tau, B(t)\}\,dt - 2\int_0^\tau \Phi_2\{\tau - t, B(t)\}\,dt \quad . \quad \ldots(287)$$

When this equation is differentiated and account is taken of the relations

$$\frac{\partial}{\partial x}\Phi_{n+1}(x, B) = -\Phi_n(x, B), \quad \Psi(B) = \Phi_2(0, B),$$

(286) is again obtained. On inserting (274), $\tau^\star = \infty$, into (273), we find the relation

$$\int_0^\infty \Phi_3\{|\tau - t|,\ B(t)\}\,dt = \frac{F}{2}\tau + \text{const},$$

thus yielding
$$F = \lim_{\tau=\infty} \frac{2}{\tau}\int_0^\infty \Phi_3\{|\tau - t|, B(t)\}\,dt.$$

This relation might be used in order to determine the parameter in $B(\tau, c)$, when F is given. Differentiation of this relation leads again to (287).

§ 33. ABSORPTION AND SCATTERING SCHWARZSCHILD'S INTEGRAL EQUATION

When absorption and scattering are simultaneously to be taken into account, the determination of the radiation field becomes an exceedingly difficult problem. It was pointed out in § 4 that this general problem leads to a complicated non-linear integral equation of B. In order to derive this equation we must first, by means of (16) and (16′), express the Ergiebigkeit J_ν in terms of the emission $\eta_\nu = \alpha_\nu B_\nu$. This present section is devoted to this partial problem.

We suppose scattering to be uniform, $\gamma \equiv 1$. Since the above problem concerns only a definite frequency ν, we replace, for the sake of clearness, the affix ν by a dash. $B_\nu = B'$, $\sigma_\nu = \sigma'$, . .

We introduce the total optical depth

$$\tau = \int_{-\infty}^{x} \rho(\alpha' + \sigma')\, dx$$

and set
$$\lambda = \frac{\sigma'}{\alpha' + \sigma'}, \quad 1 - \lambda = \frac{\alpha'}{\alpha' + \sigma'}$$

In the outermost layers of the sun scattering plays the chief rôle, while in the deeper layers absorption predominates,

$$\lambda \to 1, \ \tau \to 0, \quad \lambda \to 0, \ \tau \to \infty \qquad \ldots (288)$$

The incident radiation being zero, we obtain, on integrating (16) and on inserting the intensity into (16′), $\eta_\nu = \alpha_\nu B_\nu$, Schwarzschild's integral equation

$$J'(\tau) = \lambda(\tau)\, \Lambda(J')_\tau + \{1 - \lambda(\tau)\}\, B'(\tau) \qquad \ldots\ldots (289)$$

for the determination of the Ergiebigkeit, Λ being Milne's operator (52) Milne has shown that an equation of the type of (289), together with (288), holds under much more general conditions than under local thermodynamical equilibrium. We content ourselves with a few qualitative remarks on the solution. In order that (289) have a positive solution, the N-solution must be finite. We prove that, in general, the Neumann series converges

THEOREM XX. Under the hypothesis (288), the N-series of (289) converges if
$$B'(\tau) = O(e^{\alpha\tau})$$
holds for some $\alpha < 1$.

Proof First we study the homogeneous equation corresponding to (289), but with a constant $\lambda = \bar{\lambda}$ Since (230) represents the characteristic function of Milne's equation $f = \Lambda(f)$,

$$\kappa(u) = \frac{\lambda}{2u} \log \frac{1+u}{1-u}$$

will be the characteristic function of

$$f(\tau) = \bar{\lambda} \Lambda(f)_\tau. \qquad \ldots\ldots (290)$$

We know that, for $\bar{\lambda} = 1$, $u = 0$ is the only characteristic root in the strip $|s| < 1$, and that, for $\bar{\lambda} < 1$, there are precisely two roots in that strip, being real and of opposite sign. When u increases

from 0 to 1, $\kappa(u)$ increases from $\bar{\lambda}$ to $+\infty$ For an arbitrarily given positive $\beta<1$, we can, therefore, find a $\bar{\lambda}<1$ such that β becomes a characteristic root According to the theory of Chapter IV, there exists a solution of (290), satisfying

$$f(\tau) \sim e^{\beta\tau}, \qquad \ldots (291)$$

for τ large. The following choice of β is convenient for our purposes,

$$\alpha<\beta<1, \qquad \ldots (291')$$

α referring to the hypothesis of the theorem.

From (290),

$$f+c-\lambda\Lambda(f+c)=\left(1-\frac{\lambda}{\bar{\lambda}}\right)f+c\{1-\lambda\Lambda(1)\}, \quad .. (292)$$

c being a constant We observe that $\lambda=\lambda(\tau)$ is always less than one and tends to zero as $\tau\to\infty$ According to the hypothesis made about $B'(\tau)$, and according to (291) and (291′), the first term on the right of (292) will, for all large τ, be greater than $\{1-\lambda(\tau)\}B'(\tau)$ On the other hand we can, according to $\Lambda(1)<1$, choose c so large that, in the remaining finite τ-range, the right-hand side of (292) becomes greater than $(1-\lambda)B'$ This choice of c leads thus, for all τ, to

$$\bar{J}(\tau)>\lambda(\tau)\Lambda(\bar{J})_{\tau}+\{1-\lambda(\tau)\}B'(\tau), \qquad \ldots (293)$$

$\bar{J}=f+c$. $\bar{J}$ is, of course, positive when c is sufficiently large.

This proves, in the usual way, the convergence of the N-series. Denoting the partial sums of that series by $J_n(\tau)$, $J_0=0$, we have

$$J_{n+1}=\lambda\Lambda(J_n)+(1-\lambda)B'$$

On subtracting this from (293),

$$\bar{J}-J_{n+1}=\lambda\Lambda(\bar{J}-J_n).$$

The positivity of Λ implies thus $J_n<\bar{J}$ for all n and for all τ. From $J_0<J_1<J_2<\ldots$, the existence of the limit function J', solving (289), is inferred, q.e d.

It may be mentioned that the homogeneous equation resulting from (289) has no solution (except zero) provided that $\lambda(\tau)\to 0$ converges sufficiently rapidly as $\tau\to\infty$.

§ 34 MILNE'S MODEL OF A PLANETARY NEBULA IN RADIATIVE EQUILIBRIUM

There is another problem of radiative equilibrium worth mentioning because of different boundary conditions A spherical gaseous shell is illuminated by a source of light located in the centre of the shell What is the distribution of light in the shell when in radiative equilibrium? We confine our attention to the case of purely absorbing gray material, or to the formally equivalent case of monochromatic radiative equilibrium Let, for the sake of simplicity, the shell be infinitesimally thin, the optical thickness $\tau^\star$ being kept finite (see the beginning of § 5), i e. we neglect curvature.

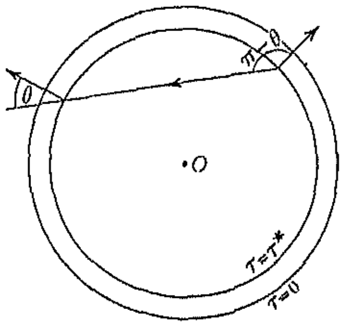
We introduce the same variables τ and θ as in the previous models of a stellar atmosphere The boundary conditions are then as follows.

(*a*) There is no radiation incident on the outer face, $\tau = 0$,

$$I(0, \theta) = 0, \quad \theta > \pi/2$$

(*b*) The inner face receives the normally incident parallel radiation of the point source O, the net flux, at $\tau = \tau^\star$, being πS.

(*c*) This, however, does not exhaust the radiation incident at the inner face It also receives radiation emergent from other points of the inner face. The radiation not being weakened while travelling through empty space, the figure shows that we have



$$I(\tau^\star, \theta) = I(\tau^\star, \pi - \theta), \quad \theta < \pi/2$$

at the inner face.

This apparently represents a new type of boundary condition. It might be regarded as a special case of

$$I(\tau^\star, \theta) = I(\tau^\star, \pi - \theta) + i(\theta); \quad \theta < \pi/2, \ldots\ldots (294)$$

$i(\theta)$, the excess of the incident over the emergent radiation, being given. When the finite size of the central star in O is taken into

account, $i(\theta)$ represents a finite function The case of parallel radiation normally incident at $\tau = \tau^\star$ must, as in § 17, be treated as a limit case †

We now derive Milne's integral equation of the problem, under the boundary conditions (a) and (294). Schwarzschild's fundamental integral equation of the plane model, with the boundary condition (a), and with any radiation incident at $\tau = \tau^\star$, was given by (64) and (64′). According to (294), the last term in (64′) splits into the summands

$$\frac{1}{4\pi}\int_{+} e^{-(\tau^\star-\tau)\sec\theta}\, I(\tau^\star, \pi-\theta)\, d\omega + \frac{1}{4\pi}\int_{+} e^{-(\tau^\star-\tau)\sec\theta}\, i(\theta)\, d\omega \quad \ldots (295)$$

The radiation in the first term comes through the shell from its outer parts,

$$I(\tau^\star, \pi-\theta) = \sec\theta \int_0^{\tau^\star} e^{-(\tau^\star-t)\sec\theta} J(t)\, dt, \quad \theta < \pi/2$$

Inserting this into (295) and using (49), $n=1$, we obtain the integral equation

$$J(\tau) = L^\star(J)_\tau + \frac{1}{4\pi}\int_{+} e^{-(\tau^\star-\tau)\sec\theta}\, i(\theta)\, d\omega, \quad \ldots (296)$$

where
$$L^\star(J)_\tau \equiv \tfrac{1}{2}\int_0^{\tau^\star} J(t)\,[E(|\tau-t|) + E(2\tau^\star-\tau-t)]\, dt \quad \ldots (296')$$

On proceeding, as in the beginning of § 17, to the limit case of normally incident radiation i of net flux πS, we obtain Milne's equation

$$J(\tau) = L^\star(J)_\tau + \frac{S}{4} e^{\tau-\tau^\star} \quad \ldots (297)$$

The net flux πF, being constant in the shell, is found from (294),

$$F = \frac{1}{\pi}\int_{+} i(\nu)\cos\theta\, d\omega.$$

In the present limit case we find $F = S$

† For a finite central star, (294) does not precisely represent the condition of the problem, because the central star covers part of the shell In the limit case, the condition is, of course, exact It will, however, be of advantage in the sequel to consider the mathematically more general condition (294)

It might be mentioned that the integral equation can easily be brought into the simpler form

$$J(\tau)=\tfrac{1}{2}\int_0^{2\tau^\star} J(t)\,E(|\tau-t|)\,dt+\frac{S}{4}e^{-|\tau^\star-\tau|},$$

$J(\tau)$ being defined in the larger interval $(0, 2\tau^\star)$ by reflection at $\tau=\tau^\star$ This shows, as before, that the solution is unique and given by the convergent N-series The same is, of course, true in the more general case (296), since this equation admits of an analogous transformation

The radiation emergent from the outer face is given by the usual formula (47), $\tau=0$ It is of interest to know how the model, S (more generally $i(\theta)$) being given once for all, behaves as the shell becomes more and more opaque ($\tau^\star\to\infty$) Milne's approximate formulae suggest that the distribution of light, in particular of the emergent radiation, becomes the same as in model Ia, $F=S$ The mere fact that the flux $F=S$, in the shell, remains the same for all $\tau^\star$, shows already that the shell cannot become dark as $\tau^\star\to\infty$ The central star is, of course, seen through the shell, but its light is dimmed by absorption (or scattering). The greater part of the light of the shell comes, therefore, from the shell itself We now give a rigorous proof of the above-mentioned fact

THEOREM XXI When $\tau^\star\to\infty$, the solution of (297) approaches the function $\tfrac{3}{4}Sf(\tau)$,

i e the Ergiebigkeit in the case Ia, $F=S$, the convergence being uniform in every finite τ-interval.

Proof. It is convenient to split the equation (297) into two simultaneous integral equations,

$$\bar{g}(\tau)=L(\bar{g})_\tau+\frac{S}{4}e^{\tau-\tau^\star}, \qquad \text{......(298)}$$

L being Schwarzschild's operator (64), and

$$h(\tau)=L^\star(h)_\tau+\tfrac{1}{2}\int_0^{\tau^\star}\bar{g}(t)\,E(2\tau^\star-\tau-t)\,dt, \qquad \text{......(298')}$$

$L^\star$ being the present operator (296'). We obviously have

$$J=\bar{g}+h; \qquad \text{......(299)}$$

$\bar{g}$, h depend, of course, upon $\tau^\star$ too. The physical meaning of (298) is that $\bar{g}$ represents the Ergiebigkeit when the radiation received from the other portions of the inner face is neglected. Finally, (298′) is the equation of the problem, where the true boundary condition (294) is taken account of, $i(\theta)$ being identified with the radiation emergent in the former case

According to the symmetry property (210) of Schwarzschild's operator, $g(\tau)=\bar{g}(\tau^\star-\tau)$ becomes the solution of

$$g(\tau)=L(g)_\tau+\frac{S}{4}e^{-\tau}. \qquad \ldots (300)$$

If in the N-series representing g all integrals are extended up to infinity instead of to $\tau^\star$, all terms obviously increase, i.e. $g(\tau)$ is smaller than the N-solution of the same equation, but with Milne's operator Λ The latter solution has, however, in § 17 been shown to be bounded $\bar{g}(\tau)$ is thus uniformly bounded for all $\tau^\star$. Therefore, from (298′),

$$h(\tau)<L^\star(h)_\tau+\frac{c}{4}E_2(\tau^\star-\tau), \qquad \ldots (301)$$

$c>0$ being a suitable constant.

We now make a crude comparison with model Ia. The theory of model Ia shows that there

$$I(\tau^\star,\theta)-I(\tau^\star,\pi-\theta)>c'\cos\theta,$$

$\theta<\pi/2$, c' being a suitable positive constant Inserting this into (296), J being identified with the function $f(\tau)$ of Problem Ia, we get

$$f(\tau)>L^\star(f)_\tau+\frac{c'}{2}E_3(\tau^\star-\tau) \qquad \ldots (302)$$

(301) implies that $h(\tau)$ lies below the N-series corresponding to the last term in (301) Similarly, $f(\tau)$ lies above the N-series corresponding to the last term in (301) From $E_2<2E_3$ we find, therefore,

$$h(\tau)<\frac{c}{c'}f(\tau),$$

thus yielding, altogether, an inequality

$$J(\tau)<Sa(\tau+1), \qquad \ldots (303)$$

a being independent of $\tau^\star$.

(303) will now be used to show the uniform continuity of $J(\tau)$ for all $\tau^\star$. Let us first fix an (otherwise arbitrary) interval $0 \leqslant \tau \leqslant k$. We choose $\tau^\star > k$ and express, according to (296) and (297), the difference

$$J(\tau'') - J(\tau'), \quad 0 \leqq \tau' \leqq \tau'' \leqq k,$$

in terms of integrals. Arranging these integrals in a convenient way, we find

$$\begin{aligned} |J(\tau'') - J(\tau')| < \frac{a}{2}\int_0^\infty (t+1)\,|E(|\tau''-t|) - E(|\tau'-t|)|\,dt \\ + \frac{a}{2}\int_0^{\tau^\star} (t+1)\,|E(2\tau^\star - \tau'' - t) - E(2\tau^\star - \tau' - t)|\,dt \\ + S\frac{e^{-\tau^\star}}{4}\,|e^{\tau'} - e^{\tau''}|. \end{aligned}$$

The second term is readily seen to be smaller than

$$(\tau'' - \tau')\frac{e^{k-\tau^\star}}{\tau^\star - k}\,\frac{(\tau^\star+1)^2}{2}.$$

The above inequality signifies, therefore, that in an arbitrarily given τ-interval $(0, k)$, the functions $J(\tau)$ are uniformly continuous, for all $\tau^\star > k+1$.

According to Arzela's selection theorem, any sequence of functions $J(\tau, \tau^\star)$, taken for an arbitrary sequence of numbers $\tau_\nu^\star \to \infty$, possesses a subsequence that converges uniformly in every finite τ-interval. Let $\tilde{J}(\tau)$ be the limit function of such a subsequence. It is on account of the uniform inequality (303), that, in (297), the integration and limit process can be interchanged, thus yielding

$$\tilde{J}(\tau) = \Lambda(\tilde{J})_\tau, \quad \tilde{J} \geqq 0,$$

Λ being Milne's operator. Theorem V shows, therefore, that $\tilde{J}(\tau)$ is a constant multiple of $f(\tau)$. The constant of proportionality is easily determined by means of the fact that the flux constant is always S. On applying the auxiliary theorem of § 13 to (297) and to the relation

$$1 = L^\star(1)_\tau + \tfrac{1}{2}E_2(\tau) + \tfrac{1}{2}E_2(2\tau^\star - \tau),$$

we find $\displaystyle 2\int_0^{\tau^\star} J(\tau)\,[E_2(\tau) + E_2(2\tau^\star - \tau)]\,dt = S(1 - e^{-\tau^\star}).$

According to (303), we may again proceed to the limit $\tau^\star \to \infty$ under the integral sign,

$$2\int_0^\infty \tilde{J}(\tau)\, E_2(\tau) = S,$$

thus yielding $\tilde{J}(\tau) = \frac{3}{4} S f(\tau)$. Since, thus, all converging subsequences of functions J, $\tau^\star \to \infty$, have the same limit function, we infer that $J(\tau, \tau^\star)$ converges towards the mentioned limit function as $\tau^\star \to \infty$, the convergence being uniform in every finite τ-interval, q.e.d.

When the radiation from other portions of the inner face is neglected, i.e. when (*b*) is the only boundary condition at $\tau = \tau^\star$, the integral equation is (298), $\bar{g}(\tau)$ being the Eigiebigkeit. It is, in this case, physically obvious that the shell becomes completely dark as $\tau^\star \to \infty$, i.e. that

$$\bar{g} \to 0, \quad \tau^\star \to \infty \qquad \dots \quad (304)$$

This can be rigorously proved as follows. Applying the auxiliary theorem of § 13 to (208) and (300), we obtain

$$\frac{S}{2}\int_0^{\tau^\star} u(t)\, e^{-t}\, dt = \int_0^{\tau^\star} g(t)\, E_2(\tau^\star - t)\, dt = \int_0^{\tau^\star} \bar{g}(t)\, E_2(t)\, dt.$$

Knowing from § 22 that $u < 1$ and that $u \to 0$ as $\tau^\star \to \infty$, we infer that the last integral in this equation tends to zero as $\tau^\star \to \infty$.

In (298), the last term tends to zero as $\tau^\star \to \infty$. As to the integral representing $L(\bar{g})$, we split the integrand into the two factors

$$(\sqrt{\bar{g} E_2})\left(\sqrt{\bar{g}}\,\frac{E(|\tau - t|)}{\sqrt{E_2(t)}}\right).$$

By application of Schwarz's inequality,

$$L(\bar{g})_\tau{}^2 < \tfrac{1}{4}\int_0^{\tau^\star} \bar{g} E_2\, dt \int_0^{\tau^\star} \bar{g}(t)\, \frac{E^2(|\tau - t|)}{E_2(t)}\, dt.$$

The first factor has been shown to tend to zero as $\tau^\star \to \infty$. According to the boundedness of $\bar{g}$, the second factor is less than a constant times

$$\int_0^\infty \frac{E^2(|\tau - t|)}{E_2(t)}\, dt < \infty.$$

(298) shows thus that (304) is true.

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It should be emphasized that this bibliography refers only to mathematical contributions to the problems in question, and to the original memoirs concerning the astronomical and geophysical applications. A comprehensive exposition of the theory of radiative equilibrium and of the approximate solution of the problems is given in Milne (1).

To §§ 1–4. The theoretical foundations of the theory of radi-

ative equilibrium are due to Schwarzschild (1), (2). He made the first fundamental applications to the sun's atmosphere (law of darkening on the sun's disk, origin of the Fraunhofer lines in the spectrum) To the interior of a star, the theory was first applied by Eddington The mathematical problems arising from the stellar interior are, however, not taken up in this tract.

It is necessary to say a few words about the customary introduction of the fundamental quantities of the radiation field The use of several differentials ds, $d\sigma$, $d\omega$, dm, dt, especially in the definition of intensity, is rather troublesome From the point of view of the mathematician, it lacks both rigour and beauty The entire subject ought to be treated anew, and the fundamental equations be derived in a rigorous way, the main tool being the general theory of measure (in the sense of Lebesgue-Radon).

To § 5. The purely absorbing gray model was first introduced by Schwarzschild (1), while the model of a purely scattering atmosphere is due to Schuster. The radiation field was originally replaced by two antiparallel streams (Schuster-Schwarzschild-approximation) The Schuster model was rigorously set up and treated in Schwarzschild's fundamental memoir (2) (case of uniform scattering). Finally, the rigorous equations for the absorbing gray atmosphere were approximately solved by Milne, who also studied the spectral distribution of the emergent light (cf Milne (1))

To § 7 The reduction to integral equations of boundary value problems of the elementary theory of radiation is due to Hilbert, who used it in his attempt to prove Kirchhoff's laws. The fundamental integral equation (289) for the Ergiebigkeit, obtained from (16) and (16′) in the plane case, is due to Schwarzschild (2). The integral equations (53) and (54) of Problems I, II have been given by Milne (1), (2) Milne established (53) originally in a different form, which is obtained easily by partial integration of the integral $\Lambda(J)_\tau$ On writing

$$C(\tau)=\int_0^\tau J(t)\,dt, \qquad \ldots(305)$$

we find

$$\Lambda (J)_\tau \equiv \int_0^\tau \frac{C(\tau+t)-C(\tau-t)}{2t}\, e^{-t} dt + \int_\tau^\infty \frac{C(t+\tau)}{2t}\, e^{-t} dt \quad . . \quad (306)$$

The importance of the positivity of the kernels has already been realized by Hilbert and Schwarzschild

To § 9. This method was first given in the author's paper (1). The existence of a solution could, of course, also be obtained from Schwarzschild's theorem about the solution of (64′), the last term being (66) Schwarzschild proved that the solution tends towards a limit function as $\tau^\star \to \infty$. Since (cf Theorem XVI) for every $\tau^\star$ the solution lies below $\tau+1$ and above τ, it could easily be seen that limit process, $\tau^\star \to \infty$, and integration can be interchanged in Schwarzschild's integral equation The limit function lies thus between τ and $\tau+1$ and satisfies Milne's equation (cf Kostitzin, Hopf (1)) In view of the central significance of Model I*a*, however, a direct treatment seems preferable The relation $c=\frac{4}{3}F$ was independently proved by Bronstein (1) and by the author (6) The simpler proof given here is the author's.

To § 10. The uniqueness was proved by the author (2).

To §§ 11, 12. Concerning the more general equation, treated there, see also Hopf (3).

To § 14. Formula (154) for the boundary temperature was independently proved by Bronstein (2) and by the author (6). The common root of both proofs is the general formula (151). This formula is equivalent with the following fact, proved by Ambarzumian in the case of Milne's equation. The resolvent kernel $K(\tau, t)$ of the given kernel $H(|\tau-t|)$ satisfies the relation $K(0, t)=f'(t)/f_0$. Concerning Theorem X see the author's paper (8).

To §§ 15, 16. The increasing of $q(\tau)$ was proved by the author (7). The equation (176) is due to Bronstein (3). The numerical results show that, at least for small and large τ, Eddington's

approximation of $q(\tau)$ is the most accurate one (cf. Milne (1), Table I).

To §17 Model I*b*, for parallel incident radiation, was considered by Milne and Eddington. The case $F=0$ applies, according to Milne (2), to the upper air in the earth's atmosphere, while the case $F>0$ is realized in the case of close binary stars (reflection effect) For the approximate solution see Milne (1).

To §21 Concerning Arzela's selection theorem, used also in §(35), see Courant-Hilbert, *Methoden der Mathematischen Physik*, Berlin.

To §§21–23. Theorem XVI is, in the case of uniform scattering, due to Schwarzschild (2), as well as the convergence towards a limit model, expressed in Theorem XVII. The Schuster-Schwarzschild approximate formula is

$$J = I^{\star}\frac{\tau+\frac{1}{2}}{\tau^{\star}+1},$$

i e the mean of Schwarzschild's limits Schwarzschild computed the correction on replacing the integral equation by a system of linear algebraic equations The approximate expression for $J(\tau)$, given in (217′), can numerically be computed by using a very good approximation of $q(\tau)$, for instance Eddington's (Milne (1)), which probably differs less than 0·5 per cent. from the true $q(\tau)$ A comprehensive account of the Schuster-Schwarzschild problem is given in Milne (1), §14.

To §§24–30. The solution by means of Fourier-Laplace integrals was given by Norbert Wiener and the author. The explicit, though very complex, formula for the law of darkening, resulting from this theory, is of main interest in astrophysics. It must, however, be mentioned that this method, though solving Problem I*a* explicitly, has but a limited applicability, as concerns many other problems of radiative equilibrium. The comparison methods, resting on the positivity of the kernel, furnish a simpler and more general access to a physically satisfactory (though not explicit) solution.

The simpler equation (226′),

$$B(\tau)=\tfrac{1}{2}\int_{-\infty}^{+\infty} B(t)\,E(|\tau-t|)\,dt, \qquad \ldots \quad (307)$$

can be interpreted as the approximate form at great depth of Milne's exact equation. The only characteristic roots being $u=0$ (double), $B(\tau)=a\tau+b$ is suggested to be the only solutions of the type (236). On using (305), $J=B$, the above equation becomes

$$C'(\tau)=\int_0^{\infty}\frac{C(\tau+t)-C(\tau-t)}{2t}e^{-t}dt \qquad . . (307')$$

This equation has been studied by Littlewood and by Hardy and Titchmarsh. An account of their results is given in Milne (1) The latter two authors have proved that the only continuously differentiable solutions of type (236) are given by

$$C(\tau)=a\tau^2+b\tau+c.$$

The more general equation, obtained from (307′) on replacing e^{-} by an arbitrary function vanishing like an exponential function as $t\to\infty$, has been treated by the author (4).

The proof of Theorem X, § 29, was given by the author (8)

To § 34. About the relation between radiation and temperature, in absence of local thermodynamical equilibrium, see Milne (1), § 19 Our λ is called there $1/1+\eta$.

To § 35 The 'planetary nebula' problem was treated in Milne (3) Reference to the former work by Jeans and Gerasimovich is found at the end of this paper.

LIST OF SOME FORMULAE OF PHYSICAL INTEREST

Pure absorption. Gray material in strict radiative and local thermodynamical equilibrium. *Model Ia.*

$$\beta(\tau) = \frac{\sigma}{\pi} T_\tau^4 = \tfrac{3}{4} F \{\tau + q(\tau)\}, \quad \frac{1}{\sqrt{3}} \leqq q(\tau) < q_\infty,$$

πF being the net flux, T_τ being the temperature at the optical depth τ.

$$T_0^4 = \frac{\sqrt{3}}{4} T_e^4,$$

T_0 being the surface, T_e the effective temperature.

$$q_\infty = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \left(\frac{3}{\sin^2\phi} - \frac{\tan\phi}{\phi - \tan\phi} \right) d\phi$$

The function

$$\sqrt{3}\Phi(u) = \left(1 + \frac{1}{u}\right) \exp\left\{ -\frac{u}{\pi} \int_0^{\frac{\pi}{2}} \frac{\log[(1 - \phi \operatorname{cotan}\phi)/\sin^2\phi]}{u^2 \cos^2\phi + \sin^2\phi} \, d\phi \right\}$$

can once for all be tabulated. The law of darkening is (see end of § 25)

$$I(0, \theta) = \tfrac{3}{4} F \Phi(\sec\theta)$$

This holds also in the absence of local thermodynamical equilibrium

Model Ib: Parallel incident radiation of normal net flux πS, the angle of incidence being θ'. The absorption coefficient for the incident radiation is supposed to be a constant fraction n of the general absorption coefficient. The following formulae give the upper and inner limiting temperature T_0, T_∞ $\left(B = \frac{\sigma}{\pi} T^4\right)$,

$$B_0 = \frac{\sqrt{3}}{4} n S \Phi(n \sec\theta'); \quad B_\infty = \tfrac{3}{4} S \cos\theta' \Phi(n \sec\theta')$$

and

$$B_0 / B_\infty = n \sec\theta' / \sqrt{3}.$$

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